

# Localization, CP-symmetry and neutrino signals of the Dirac matter.

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The connection between the Dirac field as the field of matter and the spacetime metric is discussed within the framework of classical field theory. Polarization structure of the Dirac field is shown to be rich enough to determine the spacetime metric locally and to explain the emergence of observed matter as localized waveforms. The localization of the waveforms is explained as the result of the local time slowdown and the Lorentz contraction as a dynamic re-shaping of the waveforms in the course of their acceleration. A definition of mass as a limiting curvature of the spinor-induced metric is proposed. A view of the vacuum as a uniformly distributed unit invariant density of the Dirac field with an explicitly preserved invariance of the light cone is brought forward. Qualitative explanation of the observed charge asymmetry as the consequence of the dynamics of localization is given. The origin of the  $CP$ -violation is associated with the loss of the Poincaré invariance due to localization. Neutrinos are identified with the signals emitted in the abrupt processes of creation or decay of localized objects and the concept of the Majorana neutrino is revisited. The wave equation for the classical pion field is derived from the Dirac equation. Its connection with stresses, mass and charge fluxes in localized waveforms of the Dirac field is traced. Some implications of the finite size of colliding objects for high-energy processes are discussed. A possible difference between the lifetimes and gyromagnetic ratios for positive and negative charges is predicted. A hypothesis that known internal degrees of freedom are the local spacetime (angular) coordinates that have no precise counterparts in Riemannian geometry is proposed.

## I. INTRODUCTION

The purpose of this paper is to show that the classical theory of the Dirac field, considered as a primary form of matter can explain those important properties of the observed matter which so far remain a mystery when viewed from the perspective of quantum field theory. In the first place, these properties are localization of elementary objects and the origin of their mass and finite size. Another, not less intriguing question is the origin of the observed charge asymmetry of normal matter – we find only small, heavy, positively charged protons (nuclei) and light, negatively charged, poorly localized electrons as the only stable particles around us. It seems that a possibility to answer these big questions has been overlooked at the early stage of field theory.

An idea that the fields  $\psi(x)$  of matter themselves can immediately define the metric tensor  $g_{\mu\nu}(x)$  was brought forward by Wigner [1] and Sakharov [2]. From the physics perspective, this idea is extremely sound; coordinates can be measured only through positions and shapes of material bodies. [In quoted works, tensor indices of  $g_{\mu\nu}$  were due to the derivatives  $\partial_\mu\psi(x)$ .] It appears that the Dirac field builds up the metric of spacetime without resorting to *ad hoc* derivatives and it does this in such a way that the time slows down in the domains of magnified invariant density. This observation alone leads to a natural and startlingly elegant answer to these big questions. Merely in the spirit of Huygens principle, this fact leads to self-localization of the Dirac wave forms into small,

heavy, positively charged objects of finite size while leaving a negatively charged fraction of matter in the form of an agile substance surrounding small and heavy objects. It also changes the image of the Dirac sea as the vacuum – a uniformly distributed unit invariant density  $\mathcal{R}$  is identified with  $g_{00} = 1/\mathcal{R}^2 = 1$  and replaces a continuum of oscillators with an unbound energy spectrum. The local time slowdown *dynamically* generates the physical difference between the charges of opposite sign. It is eventually translated into non-geometric nature of the discrete  $P$ - and  $T$ - reflections and into physical difference between left and right, thus becoming the underlying reason for the phenomena associated with the  $CP$ -violation.

From the perspective of the present work, the Dirac field is important, not as a special representation of the Lorentz group, but as a field that accurately describes the hydrogen atom. Lagrangian formalism is not used and no symmetry is assumed *a priori*. The main focus is on the possibility of deriving the most important properties of observed stable matter and its motion starting from the basic properties of the Dirac field and its equation of motion. No significant attempt to develop a formal perturbation theory that could have dealt with the finite size of particles as the *in*- and *out*- states of the quantum scattering process has been made so far. Once localized wave forms are found they immediately can be used as a basis for second quantization and their fields can serve as Heisenberg operators. The best prospect of this study is connected with the possibility to bridge the gap between classical point-like particles and the plane waves of the quantum theory of scattering.

The logic of the present work can be outlined as follows:

The stage is set in Sec.II A, beginning from a review

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of well-known properties of the bilinear forms of the Dirac field with emphasis on their purely algebraic origin. These forms are empirically verified to be affine Lorentz tensors at a generic point and they are further used to build a quadruple of orthogonal Lorentz unit vectors (tetrad). The possibility of treating these unit vectors as the tangent vectors of the coordinate lines of a usual holonomic coordinate system and thus to define the Riemannian metric as a descendant of the Dirac field is considered in Sec.II B. It appears that certain conditions of integrability should be met and that these conditions are controlled by the Dirac equation.

The rules of differential calculus for the Dirac field in curved spacetime are reviewed in Appendix A, mostly following V. Fock [3]. A greater generality than in [3] is intentionally admitted – and there was no possibility to truncate it later. Sec.III thoroughly investigates if various differential identities, derived from the Dirac equation, can be put in the form of tensor equations, thus being independent of a particular choice of coordinate system. For identities that involve the energy-momentum  $T^a_b$ , the conclusion is negative with the following facts firmly established: (i) The normal covariant form of the energy-momentum conservation cannot be assembled when the coordinate system is normal. (ii) The tetrad components  $T^a_b$  of energy-momentum are not the invariants of tensors. (iii) Being formally translated into coordinate form, the identity of energy-momentum balance keeps an explicit dependence on tetrad vectors. It does not reproduce the equation for a geodesic line in a given metric background. (iv) An explicit expression for the force of gravity (inertia) is derived from the constraint, which accounts for the interplay between scalar and pseudoscalar quantities in the course of a physical acceleration of a Dirac object. Only in a crude approximation of a point-like object is the standard metric expression recovered. To account for the flux of momenta in spacelike directions (pressure) in a localized object, a new stress tensor,  $P^a_b$ , is introduced and studied in the same detail in Appendix B. A connection with the theory of Nambu and Jona-Lasinio is traced. The wave equation for the pseudoscalar density (pion field) with the source that has the structure of the axial anomaly is derived in Section B 2.

An irremovable dependence on tetrad vectors prompted a detailed investigation (in Sec.IV) of the geometric properties of vector and axial currents and constraints that affect the integrability of differential equations for their lines. It is shown that for stable configurations of the Dirac field, the timelike congruence of lines of the vector current always is normal so that there always exists a system of hypersurfaces of a constant time. The key result reads as  $dt = \mathcal{R}ds_0$ , where  $\mathcal{R} = \sqrt{j^2}$  is the invariant density of the Dirac field;  $dt$  and  $ds_0$  are the intervals of the world and proper time, respectively. It immediately predicts a general trend of self-localization of the Dirac field into finite sized objects and a Lorentz contraction of accelerated

elementary objects as a physical process. Investigation of constraints connected with identities for the axial current (which, having a source, determines a radial direction) brought about another result – the maximal curvature of the 2-d surface of constant radius cannot exceed  $m$ , the mass parameter in the Dirac equation. A set of relations that connect bending of coordinate lines with the distribution of the axial current is obtained. These relations prompt a strong parallel between the local dynamics of the Dirac field and systems of inertial navigation – linear acceleration inevitably causes a precession and *vice versa*. Sec.IV C deals with the intuitively appealing (and possibly not realistic) case of normal radial coordinate, in which behavior of the angular coordinates can be studied analytically.

With the metric explicitly depending on the field of matter the Dirac equation becomes nonlinear in a unique way, which leads to self-localization as an intrinsic property of the Dirac field. Different forms of this equation along with an analysis of individual terms are the subject of Sec.V and Appendix C.

A major conjecture regarding the nature of electric charge is made in Sec.VI. The physical meaning of the discrete  $C$ -,  $P$ - and  $T$ - symmetries is reconsidered in the context of the localized objects. The origin of the  $CP$ -violation is associated with the absence of the Poincaré invariance in their interior. The residual long-range interaction between neutral objects is roughly estimated and identified as the Newton force. Maxwell equations are introduced. It is shown that a stable Dirac waveform cannot interact with its own electric field. Furthermore, two such forms cannot intersect each other in spacetime. The origin of electromagnetic radiation is explicitly traced back to the loss of simultaneity between the Dirac waveform and its Coulomb field.

We study in Sec.VII behavior of the polarization currents near characteristic surfaces,  $\mathcal{R}^2 = 0$  (where the hypersurfaces of the constant world time can be discontinuous). These surfaces are proved to be the leading fronts of the signals of the spinor field that are emitted in the course of the abrupt changes of the localized objects. These fronts carry sudden phase shift between left and right spinor components and are associated with the Dirac neutrinos. The problem of the Majorana neutrino is also revisited.

We conclude in Sec.VIII with a short list of the existing data and experiments that are in line with or can serve as the tests for our predictions.

The results of this work, if looked at as a launch-pad for further investigations, are striking in their anticipated mathematical complexity and physical transparency. It seems, however, that the former is the inevitable toll for the latter. The nonlinearity of the Dirac equation makes finding its explicit solutions a formidable task. But this nonlinearity is not artificial – no *ad hoc* nonlinear terms were added to the basic Dirac Lagrangian in order to simulate any experimentally found patterns of matter behavior, symmetry, etc. On the contrary, the discovered

generic structure corresponds to the perfectly understood phenomenon of localization, which is due to the local time slowdown, and then the loss of certain elements of spatial symmetry due to localization. Despite being genuinely nonlinear, these phenomena are so natural for any kind of wave propagation that only a minimal amount of information about the physical nature of the waves is needed to not only understand the whole picture qualitatively, but even to make semi-quantitative estimates. In the text we also outline how the existence of the pion field or how the known properties of the neutrino can be inferred from the concept of a localized Dirac object – a *waveform*.

## II. DIRAC FIELD AND RIEMANNIAN GEOMETRY.

The first attempts to bring the Dirac equation into the framework of General Relativity (GR) was made by V. Fock [3] and H. Weyl [4] in a series of papers in 1929. This study (and many other studies of that year) was in line with the basic concept of Einstein’s GR that, in the local limit (inertial reference frame), one has to reproduce the results of special relativity; it was established earlier that spinors do indeed provide a linear representation of the Lorentz group. Somewhat later, E. Cartan pointed to an insurmountable difficulty – there are no representations of the general linear group of transformations  $GL(4)$  that are similar to spinor representations of the Lorentz group of rotations. Cartan stated the following theorem, which vetoed spinors in Riemannian geometry:

*“With the geometric sense given to the word “spinor” it is impossible to introduce spinors into classical Riemannian technique; i.e., having chosen an arbitrary system of co-ordinates  $x^\mu$  for space, it is impossible to represent spinor by any finite number of components  $\psi_i$  such that  $\psi_i$  have covariant derivatives of the form  $\psi_{i;\mu} = \partial_\mu \psi_i + \Gamma_{i\mu}^j \psi_j$ , where  $\Gamma_{i\mu}^j$  are determinate functions of  $x^\mu$ .”* [5]

Of these two underscored reservations of Cartan, the first one was investigated by Ne’eman *et al* [6], who proposed to overcome the veto by resorting to the infinite-dimensional representations of the Lorentz group. The present study explores the window, which is left open by the second reservation<sup>1</sup>. As long as natural coordinates for the Dirac field are nonholonomic (also in the sense

of the theory of dynamical systems) the “connections”  $\Gamma_{i\mu}^j$  and, eventually, the metric tensor  $g_{\mu\nu}(x)$  appear to be functions of the Dirac field and not determinate functions of  $x^\mu$ .

In this section we set the stage by demonstrating that the two key issues of geometry, direction and distance, can be separated in a “physical way” by associating the field of directions with the Dirac field of matter. The Riemannian metric of the macroscopic world will then be associated with the propagation of signals.

### A. Algebraic properties of the Dirac Field.

All observables associated with the Dirac field are bilinear forms built with the aid of Dirac matrices  $\alpha^i$  and  $\beta$ , which satisfy the commutation relations ( $\alpha^a = (1, \alpha^i)$ ;  $a = 0, 1, 2, 3$ ;  $i = 1, 2, 3$ )

$$\alpha^a \beta \alpha^b + \alpha^b \beta \alpha^a = 2\beta \eta^{ab}, \quad (2.1)$$

where  $\eta^{ab} = \text{diag}(1, -1, -1, -1)$ . We begin with a review of the properties of the Dirac field  $\psi(x)$  which hold at a *point*, without a precise definition of the coordinates  $x$ . For now,  $\psi$  will stand for a column of four complex numbers  $\psi_\sigma$ .

There are sixteen linearly independent  $4 \times 4$  Hermitian matrices all of which can be constructed from the four matrices  $\alpha^i$  and  $\beta$ . The Dirac matrices,  $\rho_i$  ( $\rho_1 = \beta$ ,  $\rho_3 = -i\alpha^1\alpha^2\alpha^3$ ,  $\rho_2 = -i\beta\rho_3$ ), and  $\sigma_i = \rho_3\alpha^i$  satisfy the same commutation relations as the Pauli matrices, and all  $\sigma$  matrices commute with the  $\rho$  matrices:  $\sigma_i\sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$ ,  $\rho_a\rho_b = \delta_{ab} + i\epsilon_{abc}\rho_c$ ,  $\sigma_i\rho_a - \rho_a\sigma_i = 0$ . The matrices  $-\rho_3$  and  $\rho_1$  are commonly known as  $\gamma^5$  and  $\gamma^0$ , respectively<sup>2</sup>. Below, these matrices are used in the spinor representation,

$$\alpha_i = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 0 & -i \cdot \mathbf{1} \\ i \cdot \mathbf{1} & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

where  $\tau_i$  are the  $2 \times 2$  Pauli matrices. Employing the Dirac matrices, we can define the four components of the “vector current”,  $j^a = \psi^\dagger \alpha^a \psi \equiv \bar{\psi} \gamma^a \psi$ , the four components of the “axial current”,  $\mathcal{J}^a = \psi^\dagger \rho_3 \alpha^a \psi \equiv \bar{\psi} \gamma^5 \gamma^a \psi$ , the “scalar”  $\mathcal{S} = \psi^\dagger \rho_1 \psi \equiv \bar{\psi} \psi$  and “pseudoscalar”  $\mathcal{P} = \psi^\dagger \rho_2 \psi \equiv -i \bar{\psi} \gamma^5 \psi$ , and the six components of the skew-symmetric “tensor”  $\mathcal{M}^{ab} = (i/2) \psi^\dagger [\alpha^a \rho_1 \alpha^b - \alpha^b \rho_1 \alpha^a] \psi$ . The similarity of these quantities to the Lorentz tensors can be verified in a purely algebraic way. Indeed,

<sup>1</sup> The Cartan’s theorem has been either ignored or criticized in mathematical literature. An additional physically motivated and not restrictive requirement that, in order to support the spinor structure, the Riemannian manifold must be orientable and simply connected [7] seemingly resolved the issue. The second reservation has never been noticed. The author realized its importance only after discovering an anomaly in computation of the covariant derivatives of bilinear forms of the Dirac field, Eqs. (3.8) and (3.11).

<sup>2</sup> We consciously refrain from using the anti-hermitian matrices  $\gamma^i = \rho_1 \alpha^i$  and the Pauli-conjugated spinors  $\bar{\psi} = \psi^\dagger \rho_1$ . In their terms, the formulae of parallel transport (see Appendix A) would be much less transparent and unnecessarily complicated.

if the Dirac field  $\psi$  is transformed by means of a substitution  $\psi \rightarrow S\psi$  (or the matrices are transformed as  $\alpha^a \rightarrow S^+ \alpha^a S$ , etc.), with the matrix  $S$  depending on four complex parameters,

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^+)^{-1} \end{pmatrix}; \quad \lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$$\det|\lambda| \equiv \alpha\delta - \beta\gamma = 1, \quad (2.2)$$

then the components of  $j^a$ ,  $\mathcal{J}^a$ ,  $\mathcal{S}$ ,  $\mathcal{P}$  and  $\mathcal{M}^{ab}$  experience a four-dimensional Lorentz rotation at angles which are uniquely determined by these parameters. For example, if we take  $\alpha = e^{-i\phi/2}$ ,  $\beta = \gamma = 0$ , and  $\delta = e^{i\phi/2}$ , then the transformation  $S = e^{-i\phi\sigma_3/2}$  is unitary and the components of  $j'^a = \psi^+ S^+ \alpha^a S \psi$  are

$$\begin{aligned} j'^0 &= j^0, & j'^1 &= j^1 \cos \phi - j^2 \sin \phi, \\ j'^2 &= j^1 \sin \phi + j^2 \cos \phi, & j'^3 &= j^3, \end{aligned} \quad (2.3)$$

which corresponds to the rotation of the vector  $j^a$  at an angle  $\phi$  around the axis “3”. In exactly the same way, if we take  $\alpha = e^{-\eta/2}$ ,  $\beta = \gamma = 0$ , and  $\delta = e^{\eta/2}$  then  $S = e^{-\eta\alpha_3/2}$ ; the components of  $j'^a$  will be

$$\begin{aligned} j'^0 &= j^0 \cosh \eta - j^3 \sinh \eta, & j'^1 &= j^1, \\ j'^2 &= j^2, & j'^3 &= -j^0 \sinh \eta + j^3 \cosh \eta. \end{aligned} \quad (2.4)$$

This transformation of  $\psi$  corresponds to a Lorentz boost in the “third” direction and is *not unitary*. Similar correct relations are immediately verified for the scalars  $\mathcal{S}'$  and  $\mathcal{P}'$ , the vector  $\mathcal{J}'^a = \psi^+ S^+ \rho_3 \alpha^a S \psi$ , etc. Therefore, for example, the quantities

$$\begin{aligned} j^2 &= \eta_{ab} j^a j^b = j_0^2 - \vec{j}^2, & \mathcal{J}^2 &= \eta_{ab} \mathcal{J}^a \mathcal{J}^b, \\ j \cdot \mathcal{J} &= \eta_{ab} j^a \mathcal{J}^b = j_0 \mathcal{J}_0 - \vec{j} \cdot \vec{\mathcal{J}} \end{aligned} \quad (2.5)$$

are invariants of the  $\psi \rightarrow S\psi$  transformations. Notably, the Minkowski signature matrix of Eq.(2.1),  $\eta^{ab} \equiv \eta_{(a)} \delta_b^a$ , and its inverse  $\eta_{ab}$ ,  $\eta^{ab} \eta_{bc} = \delta_c^a$ , came up here in a purely algebraic way, and we will use it right away in order to preserve the usual convention about contraction of repeated upper and lower indices. It is also just an algebraic exercise to check that  $\mathcal{R}^2 \equiv j^a j_a = -\mathcal{J}^a \mathcal{J}_a = \mathcal{S}^2 + \mathcal{P}^2 \geq 0$  and that  $j \cdot \mathcal{J} = 0$ . When  $\mathcal{R}^2 > 0$ , the latter relation means that if the vector current of the transformed field is of the form  $j^a = (\mathcal{R}, \vec{0})$  then the axial current can only be of the form  $\mathcal{J}^a = (0, \vec{\mathcal{J}})$  and that it can be further “rotated” to  $\mathcal{J}^a = (0, 0, 0, \pm \mathcal{R})$ . Therefore, the vectors  $e_{(0)}^a = j^a/\mathcal{R}$  and  $e_{(3)}^a = \mathcal{J}^a/\mathcal{R}$  are the orthogonal timelike and spacelike unit vectors, respectively, and they can be reduced to  $e_{(0)}^a \doteq \delta_0^a$  and  $e_{(3)}^a \doteq \delta_3^a$ . (In what follows, the symbol  $(\doteq)$  is used in equations that imply such a particular reduction.)

The components of the tensor  $\mathcal{M}^{ab}$  and its dual  $\mathcal{M}^{*ab} = (1/2)\epsilon^{abcd}\mathcal{M}_{cd}$  are

$$\begin{aligned} \mathcal{M}^{0i} &= K_i = \psi^+ \rho_2 \sigma_i \psi, & \mathcal{M}^{*ij} &= \epsilon^{0ijm} K_m \\ \mathcal{M}^{*0i} &= L_i = \psi^+ \rho_1 \sigma_i \psi, & \mathcal{M}^{ij} &= \epsilon^{0ijm} L_m. \end{aligned} \quad (2.6)$$

Because  $\mathcal{M}^{ab} \mathcal{M}_{bc}^* = (\vec{L} \cdot \vec{K}) \delta_c^a$ , these two tensors can be used to build two couples of vectors which are spacelike, orthogonal to  $e_{(0)}^a$  and  $e_{(3)}^a$  and to each other,

$$\begin{aligned} E_c &= j^a \mathcal{M}_{ab} [\mathcal{R}^2 \delta_c^b + \mathcal{J}^b \mathcal{J}_c] \doteq \mathcal{R}^3(0, K_1, K_2, 0), \\ \vec{E}_c &= \mathcal{J}^a \mathcal{M}_{ab} [\mathcal{R}^2 \delta_c^b - j^b j_c] \doteq \mathcal{R}^3(0, K_2, -K_1, 0), \\ H_c &= \mathcal{J}^a \mathcal{M}_{ab} [\mathcal{R}^2 \delta_c^b - j^b j_c] \doteq \mathcal{R}^3(0, -L_2, L_1, 0), \\ \vec{H}_c &= j^a \mathcal{M}_{ab} [\mathcal{R}^2 \delta_c^b + \mathcal{J}^b \mathcal{J}_c] \doteq \mathcal{R}^3(0, L_1, L_2, 0). \end{aligned} \quad (2.7)$$

They can be normalized and employed as  $e_{(1)}^a$  and  $e_{(2)}^a$ .

A full set of easily verifiable *identities* between invariants of the transformations (2.2) is given by

$$\begin{aligned} \mathcal{R}^2 &\equiv j_a j^a = -\mathcal{J}_a \mathcal{J}^a = \mathcal{S}^2 + \mathcal{P}^2, & \mathcal{J}_a j^a &= 0, \\ \mathcal{S}^2 - \mathcal{P}^2 &= \vec{L}^2 - \vec{K}^2, & \mathcal{S} \mathcal{P} &= \vec{L} \cdot \vec{K}. \end{aligned} \quad (2.8)$$

The left and right vector currents,  $j_{(L)}^a = [j^a \pm \mathcal{J}^a]/2$ , are lightlike; therefore, we also have the identities,

$$j_{(L)}^a j_{(L)a} = j_{(R)}^a j_{(R)a} = 0, \quad \mathcal{R}^2 = 2j_{(L)}^a j_{(R)a}. \quad (2.9)$$

These currents always belong to the 2-d plane determined by the tetrad vectors  $e_{(0)}^\alpha$  and  $e_{(3)}^\alpha$ . The scalars allow for the following parameterizations,

$$\mathcal{S} = \mathcal{R} \cos \Upsilon, \quad \mathcal{P} = \mathcal{R} \sin \Upsilon, \quad (2.10)$$

where both  $\mathcal{R}$  and  $\Upsilon$  are functions of the Dirac field. It is important that the absolute values of  $\mathcal{S}$  and  $\mathcal{P}$  do not exceed  $\mathcal{R}$ . A similar observation is true for the second line of Eq.(2.8),

$$\begin{aligned} \mathcal{S}^2 - \mathcal{P}^2 &= \vec{L}^2 - \vec{K}^2 = \mathcal{R}^2 \cos 2\Upsilon, \\ 2\mathcal{S} \mathcal{P} &= 2\vec{L} \cdot \vec{K} = \mathcal{R}^2 \sin 2\Upsilon. \end{aligned} \quad (2.11)$$

Concluding the discussion of the algebraic properties of bilinear forms of the Dirac field, let us introduce, along with the orthogonal system  $e_{(a)}^\beta[\psi]$ , a reciprocal (in algebraic sense) system  $e_{(b)}^{(a)}[\psi]$

$$\sum_\alpha e_{(a)}^\alpha e_{\alpha}^{(b)} = \delta_a^b, \quad \sum_a e_{(a)}^\alpha e_{\beta}^{(a)} = \delta_\beta^\alpha, \quad (2.12)$$

where we assumed that  $\det|e_{(a)}^\beta| \neq 0$ . Then, a simple algebra verifies that the objects

$$g_{\alpha\beta} = \eta_{ab} e_{(a)}^\alpha e_{(b)}^\beta, \quad g^{\alpha\beta} = \eta^{ab} e_{(a)}^\alpha e_{(b)}^\beta \quad (2.13)$$

can be used to move the Greek indices up and down, for example,

$$g_{\alpha\beta} e_{(b)}^\beta = e_{(a)\alpha} e_{(b)}^{(a)} e_{(b)}^\beta = \delta_b^a e_{(a)\alpha} = e_{\alpha(b)}.$$

It is also evident that the repeated upper and lower Greek indices are contracted.

## B. Dirac currents and Riemannian geometry.

From now on, we look at the  $\psi_\sigma(x)$  as the physical Dirac field, the continuous functions of the arbitrarily parameterized points  $x^\mu = (x^0, x^1, x^2, x^3)$  of the spacetime. So far, we have verified that the algebraic structure of bilinear forms of the Dirac field naturally contains an orthogonal quadruple of unit Lorentz vectors at a generic point, thus defining spacetime directions *at that point*. In a sense, linear transformations (2.2) of the Dirac field generate a group of homogeneous linear transformations for vectors, thus associating with every point a local centered affine space. Vectors of this quadruple are thought of as smooth functions of  $\psi(x)$  and it is tempting to immediately treat it as a quadruple of the vector fields. But these transformations of the Dirac field have nothing to do with the general transformation of coordinates, which are arguments of  $\psi(x)$ . For a given fixed  $\lambda$ , we can consider  $x^\lambda = \text{const}$  as the equation of a coordinate hypersurface and the lines along which all coordinates but  $x^\lambda$  are constant as coordinate lines. Tangent vectors of these lines (which are gradients of the linear function  $\varphi(x) = x^\lambda$ ) are  $h_{(\lambda)}^\mu = \partial x^\mu / \partial x^\lambda = \delta_{(\lambda)}^\mu$ ; therefore, this coordinate system is an holonomic one, but it has no metric and there is no way to determine if its coordinate lines are orthogonal. One may replace  $x^\mu$  by smooth functions of other coordinates  $y^\mu$ ,  $x^\mu = f^\mu(y)$ , thus redefining coordinate lines and surfaces, but such a change does not alter  $\psi(x(y))$  and has nothing to do with affine Lorentz transformations (2.2). To bridge the gap between the abstract field of directions determined by the Dirac field and the given above definition of the holonomic coordinates, it is necessary to know in advance that four systems of differential equations (for the unknown  $x^\mu$ ),

$$\frac{dx^0}{e_{(a)}^0(x)} = \frac{dx^1}{e_{(a)}^1(x)} = \frac{dx^2}{e_{(a)}^2(x)} = \frac{dx^3}{e_{(a)}^3(x)} = ds^a, \quad (2.14)$$

for congruences of lines (labeled by the ordinal numbers  $(a)$ ) are solvable and thus determine a coordinate net. In other words, if these equations are integrable, then the system

$$dx^\alpha = e_{(a)}^\alpha ds_a, \quad \alpha = 0, 1, 2, 3, \quad (2.15)$$

will represent lines, which at every point  $x$  have a determinate direction  $e_{(a)}^\alpha(\psi)$ , and only one line of the congruence  $(a)$  passes through each point in spacetime. The tetrad vector  $e_{(a)}^\alpha$  will be a Lorentz vector and a coordinate vector. A change of the vector variables  $x^\mu = f^\mu(y)$  in Eq.(2.15) will result in the transformation of the differential  $dx^\mu$ ,

$$dx^\alpha = \frac{\partial x^\alpha}{\partial y^\sigma} dy^\sigma, \quad \frac{\partial x^\alpha}{\partial y^\sigma} \cdot \frac{\partial y^\sigma}{\partial x^\beta} = \delta_\beta^\alpha,$$

and the equation for the same congruence in new coordinates will read as

$$dy^\sigma = e_{(a)}^\sigma(y) ds_a, \quad e_{(a)}^\sigma(y) \stackrel{\text{def}}{=} \frac{\partial y^\sigma}{\partial x^\beta} \cdot e_{(a)}^\beta[\psi]. \quad (2.16)$$

A new element here is that tangent vectors depend on the coordinates via the coordinate dependence of the Dirac field which, in its turn, is constrained by the equations of motion. Therefore, the problem of integrability of Eqs.(2.14) - (2.16) cannot be addressed solely within Riemannian geometry; at least some properties of congruences must be controlled by the Dirac equation. For “temporal” and “radial” congruences, the Dirac equation indeed yields a set of constraints with a clear physical meaning. The properties of congruences of angular arcs (including their symmetry), in general, not only explicitly depend on particular solutions of the Dirac equation but there may even be no meaningful holonomic coordinates associated with these arcs. Nevertheless, even keeping such a difficult perspective in mind, let us consider all four tetrad vectors as contravariant vectors of Riemannian geometry.

Traditional approaches assume solving the Dirac equation in a determinate metric field  $g_{\mu\nu}(x)$  of spacetime. The polarization properties of the Dirac field prompt the opposite direction of thinking. Namely, the field  $\psi(x)$  must be the solution of the Dirac equation, which explicitly depends on a resulting metric  $g_{\mu\nu}[\psi(x)]$  given by Eqs.(2.13). In such a context, the Minkowski form of the metric in the local limit is associated not with an imaginable local inertial frame but rather with the algebraic properties of the Dirac field and (complementary to the latter) the hyperbolic character of the Dirac equation. Thus, it is possible to overcome Cartan’s veto in two major points. First, there is no arbitrary coordinates for spacetime (modulo a trivial change of variables). Second, the connections,  $\Gamma$ , are no longer determinate functions of  $x$ ; they become functions of the Dirac field. In this framework, as is shown below, the hypersurfaces of a constant temporal coordinate naturally emerge; their existence is a prerequisite for the quantization of the stable configurations of the Dirac field in curved spacetime. The proper time slows down in domains of a higher matter density, which points to self-localization as an intrinsic property of the Dirac field. This effect also clarifies the nature of electric charge and of charge asymmetry of the empirically known stable matter. Along with localized matter, there always exists a preferred system of orthogonal congruences determined by the internal polarization structure of physical objects. It can be considered as *the net of the nonholonomic coordinate system* [8] and thought of as a generalization of the Lagrangian coordinates  $s_a$  of the hydrodynamics for the case of polarized relativistic fluid. Integration of equations (2.14) together with the equations of motion amounts to finding the Eulerian coordinates  $x^\mu$ .

Let us follow the key idea of intrinsic geometry to associate tensor fields with mutual invariants of tensors and parameters (tangent vectors  $e_{(a)}^\mu(x)$ ) of a system of congruences. Furthermore, let us read Eqs.(2.12) and (2.13)

as

$$\begin{aligned} g_{\nu\mu}(x)e_{(a)}^\mu(x)e_{(b)}^\nu(x) &= \eta_{ab}, \\ \eta_{ab}e_{(a)}^\mu(x)e_{(b)}^\nu(x) &= g_{\nu\mu}(x), \end{aligned} \quad (2.17)$$

and consider *this*  $g_{\nu\mu}(x)$ , as a primary choice of the space-time metric. Only by virtue of Eqs.(2.14) can we translate the first of equations Eq.(2.17) into

$$\begin{aligned} ds^2 &= \eta_{ab}ds^a ds^b = g_{\nu\mu}(x)e_{(a)}^\mu(x)e_{(b)}^\nu(x)ds^a ds^b \\ &= g_{\nu\mu}(x)dx^\mu dx^\nu, \end{aligned} \quad (2.18)$$

and reconcile Eqs.(2.13) (inspired by the algebra of the Dirac matrices) with the measure of length postulated in Riemannian geometry.

When  $e_{(a)}^\mu$  is a vector with the law of transformation (2.16) and  $g_{\nu\mu}(x)$  is a tensor (not necessarily determining a metric) then the covariant derivative  $\nabla_\nu e_{(a)}^\mu$  with respect to  $g_{\nu\mu}$  is also a tensor [9]. Therefore, one can introduce a system of invariants (the Ricci coefficients of rotation of a system of congruences)

$$\omega_{bca} = e_{(a)}^\mu(\nabla_\mu e_{(b)}^\nu)e_{(c)\nu} = -\omega_{cba}. \quad (2.19)$$

For a given  $(c)$ , six parameters  $\omega_{abc}ds$  determine an infinitesimal rotation of the pyramid of tetrad vectors in the “plane”  $(ab)$  when the vertex of the pyramid is displaced by  $ds$  along a line of congruence  $(c)$ . Equation

$$\nabla_\mu e_{(b)\nu} = \omega_{bca}e_{(c)}^\nu e_{(a)}^\mu \quad (2.20)$$

is the inverse of (2.19). Using Eqs.(2.19) and (2.20), it is straightforward to check that if  $g_{\nu\mu}(x)$  has the form (2.17) then  $\nabla_\lambda g_{\nu\mu} = 0$ . Consequently, the vector connections  $\Gamma_{\mu\nu}^\sigma$  coincide with the Christoffel symbols of the metric  $g_{\nu\mu}$ .

In an ideal geometric world (i.e. when all four holonomic coordinates exist) the necessary conditions for integrability of Eqs.(2.20) are given by the Ricci identities [9],

$$(\nabla_\mu \nabla_\lambda - \nabla_\lambda \nabla_\mu)e_{(a)\nu} = e_{(a)}^\sigma R_{\sigma\nu\mu\lambda} \quad (2.21)$$

where  $R_{\sigma\nu\mu\lambda}$  is the Riemann curvature tensor. These equations can be cast in the form,

$$e_{(a)}^\sigma e_{(b)}^\nu e_{(c)}^\mu e_{(d)}^\lambda R_{\sigma\nu\mu\lambda} = R_{abcd}, \quad (2.22)$$

where

$$\begin{aligned} R_{abcd} &\equiv \partial_d \omega_{abc} - \partial_c \omega_{abd} \\ &+ \sum_f \eta_f [\omega_{fad} \omega_{fbc} - \omega_{fac} \omega_{fbd} + \omega_{abf} (\omega_{fcd} - \omega_{fdc})], \end{aligned} \quad (2.23)$$

is a system of invariants, which is then known as the tetrad representation of the Riemann tensor. Since at least some of the Ricci coefficients of rotation will appear to be functions of the Dirac field, this dependence

will be carried through onto the Riemann and Ricci tensors. The Einstein equations for the metric field  $g_{\nu\mu}(x)$  that describes motion of macroscopic objects may appear to be descendants of the constraints stemming from the Dirac equation.

To summarize, if in spacetime, with arbitrarily chosen holonomic coordinates,  $x^\mu$ , the Dirac field  $\psi(x)$  is defined and at each point the 16 quantities,  $e_{(a)}^\nu[\psi(x)]$ , are computed (along with the algebraically reciprocal system  $e_{(a)}^{(\nu)}[\psi(x)]$ ) then the metric  $g_{\nu\mu}(x)$  of spacetime is given by Eq.(2.17) and the interval by Eq.(2.18). This metric depends on the Dirac field and is not defined *a priori*. From the physical perspective, its existence seems to be a privilege of exceptional solutions rather than a rule.

It is important to realize that the material Dirac field defines a system of the *unit* vector fields  $e_{(a)}^\mu(x)$  – therefore, *the effect of such a matter-induced metric should be equivalent to a long-range interaction between localized objects*.

### III. DIFFERENTIAL IDENTITIES FOR TENSORS.

In order to find limitations on the metric of spacetime, which can host the localized configurations of the Dirac field, we begin with the examination of various identities that are consequences of the Dirac equation. The question is, whether differentials of various bilinear forms of the Dirac field, which are considered as the physical observables, can be translated into covariant derivatives of tensors. We use this question as a test of the roots of the discrepancies that could have led to Cartan’s veto. It appears that these discrepancies correspond to the clearly understood physical processes.

Following Fock and Weyl, we postulate that the equation of motion of the Dirac field and its conjugate are

$$\alpha^a D_a \psi = -im\rho_1 \psi, \quad \psi^+ \overleftarrow{D}_a \alpha^a = im\psi^+ \rho_1, \quad (3.1)$$

where the covariant derivative  $D_a \psi = (\partial_a - \Gamma_a)\psi$  of the Dirac field is defined in Appendix A. The object  $D_\mu \psi = e_\mu^a D_a \psi = (\partial_\mu - \Gamma_\mu)\psi$  will be used only as a symbol, since it has no clear geometrical meaning. The mass parameter in these equation is *a priori* arbitrary. Because the Dirac field has the property of self-localization, every stable localized waveform will determine the corresponding value of  $m$ .

#### A. Identities for vector and axial currents.

From the equations of motion (3.1) one immediately derives two well-known identities. One of them,

$$D_a j^a = \nabla_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} \psi^+ \alpha^\mu \psi] = 0, \quad (3.2)$$

clearly indicates conservation of the *timelike* vector (probability) current of the Dirac field, while the second one indicates that the *spacelike* axial current is not conserved,

$$D_a \mathcal{J}^a = \nabla_\mu \mathcal{J}^\mu = 2m\mathcal{P}, \quad (3.3)$$

and has the pseudoscalar density as a source. The same source (but with the opposite signs) have the lightlike left and right currents,

$$\nabla_\mu j_{(R)}^\mu = \pm m\mathcal{P}. \quad (3.4)$$

The significance of Eq.(3.3) is due to the pseudoscalar density  $\mathcal{P}$  on the r.h.s. Since  $\mathcal{P}$  is localized not less than  $\mathcal{R}$  and the vector  $\mathcal{J}^\mu$  is spacelike, the unit axial vector  $e_{(3)}^\mu$  defines the outward radial direction. The existence of such a direction is a distinct characteristic of a localized object.

### B. Flux of momenta: not tensors.

Consider now a more complicated object  $T_b^a = i\psi^+ \alpha^a D_b \psi$ , the Hermitian part of which is normally regarded as the energy-momentum tensor of the Dirac field. Its components are interpreted as the flux of components  $iD_b$  of the momentum in the direction of congruence of lines of the vector current  $j^a$ . Because the vector current is *timelike*, this tensor is well-suited to describe the flux of momenta carried by massive particles. When spinor field

is a solution of the Dirac equation (3.1) the Lagrangian  $L_D$  of the Dirac field equals to zero and  $T_b^a$  does not have a diagonal term,  $-L_D \delta_b^a$ , which could have been responsible for the flux of momentum in the spacelike direction (e.g., the pressure). Since the spacelike radial direction is controlled by the axial current, we are led to consider another object, the stress tensor  $P_b^a = i\psi^+ \rho_3 \alpha^a D_b \psi$ , which accounts for the flux of momenta in the radial direction. For stable localized wave forms, there must be no flux of any observables in the spacelike outward direction. However, if we decide to investigate a particle's Lorentz contraction as a dynamic process or the decay of a long-lived waveform (considered as a particle), then we are led to consider the spacelike flux of momenta due to the "phase shifts" inside the wave form. Regardless of how adequate this intuitive physical interpretation of  $T_b^a$  or  $P_b^a$  is, or even without any physical interpretation, they both can be used to derive various useful identities, which allow one to compute the rotation coefficients  $\omega_{abc}$  as functions of the Dirac field and thus constrain the possible metric (2.18). In this section, we study  $T_b^a$  in detail. The stress tensor  $P_b^a$  is studied in Appendix B along the same guidelines

The reader should not be confused by how the standard name "energy-momentum tensor" is used. In the context of the present work, the invariants  $T_b^a$  and  $P_b^a$  are the auxiliary objects. We are interested only in identities that can be derived from the Dirac equation in tetrad form and then translated, if possible, into the tensor form. Only Hermitian part of these objects enters the equations that allow for a physical interpretation.

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One would expect that the absolute differential of  $T_{ab}$ , being computed according to the Leibnitz rule, will be as follows,

$$D_c T_{ab} = \partial_c T_{ab} - \omega_{adc} T_{db} - \omega_{bdc} T_{ad} \equiv \nabla_c T_{ab}. \quad (3.5)$$

If this expectation turns out to be justified then the usual covariant derivative will be immediately reproduced as

$$\partial_\lambda T_{\sigma\mu} - \Gamma_{\sigma\lambda}^\nu T_{\nu\mu} - \Gamma_{\mu\lambda}^\nu T_{\sigma\nu} = e_\lambda^c e_\sigma^a e_\mu^b \nabla_c T_{ab} = \nabla_\lambda T_{\sigma\mu}. \quad (3.6)$$

Contrary to the expectation of (3.5), the answer reads

$$D_c[\psi^+ \alpha^a \vec{D}_b \psi] = \partial_c[\psi^+ \alpha^a \vec{D}_b \psi] - \psi^+ [\Gamma_c^+ \alpha^a + \alpha^a \Gamma_c] \vec{D}_b \psi = \partial_c[\psi^+ \alpha^a \vec{D}_b \psi] - \omega_{adc} \psi^+ \alpha^a \vec{D}_b \psi, \quad (3.7)$$

with the last term of Eq.(3.5) missing, and no hope to recover the full *geometric* expression (3.6) of the covariant derivative of the tensor! The  $\vec{D}_b \psi$  behaves as a scalar and not as a vector! [If calculations were carried out in coordinate representation then the last term in Eq.(3.6) would be lost without possibility to recover the full tetrad expression (3.5). A similar abnormal pattern is repeated in Eqs.(3.8), (3.11) and (B.1), (B.2), (B.7) below.]

Contracting indices  $a$  and  $c$  we arrive at the expression,

$$D_a[\psi^+ \alpha^a \vec{D}_b \psi] = \partial_a[\psi^+ \alpha^a \vec{D}_b \psi] + \omega_{acc} \psi^+ \alpha^a \vec{D}_b \psi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{-g} e_{(a)}^\nu (\psi^+ \alpha^a \vec{D}_b \psi) \right], \quad (3.8)$$

which is exactly what one may wish to have as the l.h.s. of a conservation law for the energy-momentum of the Dirac field. The missing term is exactly the one that does not let one interpret equations like  $\nabla_\sigma T_\mu^\sigma = 0$  as conservation of anything. However, at the moment, a covariance can not yet be explicitly visible; it may occur that the r.h.s. of an expected conservation law, which must be determined using the equations of motion, recovers the covariance of the

resulting identity as a whole. This appears to be the case; therefore, the second reservation of the aforementioned Cartan's theorem is important. Unlike the directions of fluxes, which are associated with the matrices  $\alpha^\mu$ , the directions of the derivatives (covariant vectors) are determined dynamically, they are controlled by the equations of motion.

Let us first rewrite the l.h.s. of (3.8) as

$$D_a[\psi^+ \alpha^a \vec{D}_b \psi] = \psi^+ \alpha^a [\vec{D}_a \vec{D}_b - \vec{D}_b \vec{D}_a] \psi + \psi^+ \overleftarrow{D}_a^+ \alpha^a \vec{D}_b \psi + D_b(\psi^+ \alpha^a \vec{D}_a \psi) - \psi^+ \overleftarrow{D}_b^+ \alpha^a \vec{D}_a \psi.$$

By virtue of the equations of motion (and due to the Leibnitz rule) the last three terms exactly cancel out and the final result is

$$D_a[i\psi^+ \alpha^a \vec{D}_b \psi] = i\psi^+ \alpha^a [\vec{D}_a \vec{D}_b - \vec{D}_b \vec{D}_a] \psi. \quad (3.9)$$

Using Eqs.(A.10) and (A.11) to separate the terms with and without derivatives in the commutator and comparing with (3.8) we find that the Dirac equation yields the identity:

$$\partial_a T_{ab} - \omega_{cac} T_{ab} = \omega_{bca} T_{ac} - \omega_{acb} T_{ac} - i\psi^+ \alpha^a \mathbb{D}_{ab} \psi, \quad (3.10)$$

where the commutator  $\mathbb{D}_{ab}$  does not contain derivatives of  $\psi$ . After moving the term  $\omega_{bca} T_{ac}$  from the right to the left, the l.h.s. becomes, according to (3.5) and (3.6), the system of mutual invariants of a usual covariant divergence of the tensor  $T_\mu^\sigma$  and congruences  $e_a^\mu$ . It can be transformed either into coordinate form (3.6), which has no explicit dependence on tetrad vectors or into a non-coordinate form. Unfortunately, this coordinate independence of a fragment of identity (3.10) is useless, because there remains an abnormal term  $\omega_{cab} T_{ac}$  on the right. Being translated into a coordinate form, this term becomes  $(\nabla_\lambda e_a^\nu) e_\sigma^a T_\nu^\sigma$ . It explicitly depends on tetrad vectors (on how the coordinate lines are bending).

The abnormal term  $\omega_{cab} T_{ac}$  in Eq.(3.10) enters another identity that follows from the Dirac equation. It arises after contracting indices  $a$  and  $b$  in Eq.(3.7),

$$D_c[\psi^+ \alpha^a \vec{D}_a \psi] = \partial_c[\psi^+ \alpha^a \vec{D}_a \psi] - \omega_{abc} \psi^+ \alpha^b \vec{D}_a \psi. \quad (3.11)$$

Eq.(3.11) reveals one more inconsistency, which is similar to the one observed in Eq.(3.8). The quantity  $D_c T_a^a$  is derivative of the trace of a tensor, i.e. of a scalar. However the result has an additional term with a connection, which is one more piece of evidence that the quantities,  $T_b^a$ , are not the invariants of a tensor. By the same token, the l.h.s. of Eq.(3.11) is not a covariant derivative of a scalar.

By virtue of the Dirac equation, the first term on the r.h.s. of (3.11) becomes  $\partial_c[-im\psi^+ \rho_1 \psi]$ . Alternatively, one can immediately use the equations of motion on the l.h.s. and only then differentiate,

$$\begin{aligned} D_c[\psi^+ \alpha^a \vec{D}_a \psi] &= -im D_c[\psi^+ \rho_1 \psi] \\ &= -im \partial_c[\psi^+ \rho_1 \psi] + im \psi^+ [\Gamma_c^+ \rho_1 + \rho_1 \Gamma_c] \psi. \end{aligned} \quad (3.12)$$

Comparing the last two equations and using (A.4) we finally get

$$\omega_{acb} \cdot T_{ca} = 2mg \mathcal{P} \aleph_b. \quad (3.13)$$

Using Eq.(3.13), one can then rewrite Eq.(3.10) in a formally covariant form,

$$\nabla_\sigma T_\nu^\sigma = i\psi^+ \alpha^\mu [D_\mu, D_\nu] \psi + 2mg \mathcal{P} \aleph_\nu, \quad (3.14)$$

where  $\aleph_\nu = e_\nu^{(a)} \aleph_a$  and the commutator  $D_\mu D_\nu - D_\nu D_\mu = [D_\mu, D_\nu] = -e_\mu^a e_\nu^b \mathbb{D}_{ab}$  on the r.h.s. has no derivatives. For the sake of completeness we mention that the imaginary part of  $T_\nu^\sigma$  is a tensor; it is the covariant derivative  $(i/2) \nabla_\nu j^\sigma$ . Because the vector current is conserved, the imaginary part of Eq.(3.14) is just an identity [3],  $i \nabla_\sigma (\nabla_\nu j^\sigma) = i R_{\sigma\nu} j^\sigma$ , where  $R_{\mu\sigma}$  is the Ricci curvature (contracted Riemann tensor of curvature).

An attempt to make  $\aleph_\nu = 0$  leads to the main result of Fock's paper [3], which was derived entirely in the coordinate representation (using  $D_\mu$  as a well-defined operator) and interpreted, with the reference to the correspondence principle, as the equation of a geodesic line. Since Eqs.(3.9) and (3.13) are nothing but two identities that follow from the Dirac equation and Eq.(3.14) is their sum, there obviously is a way to derive Eq.(3.14) in one step, which was done by Fock (with a subtle inaccuracy). The possibility to set  $\aleph_a = 0$  can be viewed as evidence that the abnormal term  $\omega_{cab} T_{ac}$  is zero. Such an impression is not correct. This would require that  $\omega_{cab} = 0$ . If the second term in the r.h.s. of Eq.(3.10) is zero so is the first one. There remains nothing to move to the left in order to compile the covariant derivative of the tensor.

Consider another possibility that the Riemannian spacetime, which is hosting a Dirac field configuration (like proton) admits an orthogonal system of coordinate hypersurfaces. Then the Ricci coefficients with all different ordinal numbers vanish, i.e.  $\omega_{cab} - \omega_{cba} = 0$ , and the first two terms in the r.h.s. of identity (3.10) just cancel each other. Once again, the possibility to compile the covariant derivative of  $T_{ab}$  as a part of the conservation law is lost. We are led to conclusion that *the metric of spacetime, which is hosting the Dirac field (and thus is determined by this field) does not allow for a system of orthogonal coordinate surfaces.*

If we formally translate the remaining terms into the coordinate representation, we get, instead of (3.10) and



(3.13), two equations,

$$\partial_\sigma T_\mu^\sigma + \Gamma_{\nu\sigma}^\sigma T_\mu^\nu - (\partial_\sigma e_\mu^a) e_a^\nu T_\nu^\sigma = i\psi^+ \alpha^\mu [D_\mu D_\nu - D_\nu D_\mu] \psi, \quad (3.15)$$

$$(\nabla_\mu e_\sigma^a) e_a^\nu T_\nu^\sigma = [e_a^\nu \partial_\mu e_\sigma^a - \Gamma_{\sigma\mu}^\nu] T_\nu^\sigma = 2mg\mathcal{P}\aleph_\mu, \quad (3.16)$$

both carrying an explicit dependence on the tetrad vectors. The sum of these equations reads as,

$$\partial_\sigma T_\mu^\sigma + \Gamma_{\nu\sigma}^\sigma T_\mu^\nu - \Gamma_{\sigma\mu}^\nu T_\nu^\sigma - (\partial_\sigma e_\mu^a - \partial_\mu e_\sigma^a) e_a^\nu T_\nu^\sigma = i\psi^+ \alpha^\mu [D_\mu D_\nu - D_\nu D_\mu] \psi + 2mg\mathcal{P}\aleph_\mu,$$

where this dependence is apparently hidden because we assumed that all congruences are normal and  $\partial_\sigma e_\mu^a - \partial_\mu e_\sigma^a = e_\sigma^c (\omega_{abc} - \omega_{acb}) e_\mu^b = 0$ . This equation indeed coincides with (3.14), but only when the connection,  $\Gamma_{\sigma\mu}^\nu$ , is symmetric, which was not an *a priori* requirement. Since, in general, the Ricci coefficients are not zero and the “tensor”  $T_{\sigma\mu}$  is not symmetric (except for a plane-wave solution), we cannot argue that the r.h.s. of (3.13) must be zero for whatever reason. If, in addition to (A.1), we unconditionally required that  $\delta\mathcal{S} = \delta\mathcal{P} = 0$ , then arriving at (3.13) we would generate controversy.

An *ad hoc* choice of an *orthogonal coordinate system* (where  $\omega_{abc} = \omega_{acb}$ ) can serve only as a crude approximation. To be consistent, we have to replace  $\omega_{acb} T_{ca}$  by  $\omega_{abc} T_{ca}$  in Eqs.(3.10) and Eq.(3.13) simultaneously. Then, the latter can be rewritten as  $\omega_{bca} \cdot T_{ca} = -2mg\mathcal{P}\aleph_b$ , so that the symmetric (and only the symmetric) part of  $T_{ba}$  matters. This transmutation indicates that we implicitly employed the approximation of a material point when internal deformations, that are bringing an object into a new state of motion, are disregarded (the tidal forces are ignored). In this case, the unit vector  $e_{(0)}^\mu$  plays the role of the 4-velocity  $u^\mu$  of a small object as a whole. Since coordinate system is orthogonal, the first term in brackets in Eq.(3.16) can be dropped and we may write

$$\frac{\partial}{\partial x^\mu} [\sqrt{-g} \operatorname{Re}(T_\nu^\mu)] = e\sqrt{-g} j^\mu F_{\mu\nu}, \quad (3.17)$$

$$\Gamma_{\sigma\mu}^\nu \operatorname{Re}(T_\nu^\sigma) = -2mg\mathcal{P}\aleph_\mu, \quad (3.18)$$

which is a perfect expression for the energy-momentum conservation complemented by the constraint (3.16)<sup>3</sup>. The Lorentz force in the r.h.s. of Eq.(3.17) allows one to identify the vector  $A_a$  in the connection  $\Gamma_a$  as the tetrad components of the electromagnetic potential and  $e_j^a$  as the components of the electric current density. If the equation  $\nabla_\sigma T_\nu^\sigma = 0$  is considered as a prototype for the equation of a geodesic line (as it was conjectured in [3]) then the term  $\Gamma_{\mu\sigma}^\nu T_\nu^\sigma = \Gamma_{\sigma\mu}^\nu T_\nu^\sigma$  in it is connected

with  $\aleph_\mu$  through Eq.(3.18). Depending on the nature of the physical process, this term is responsible either for the gravitational force or for the force of inertia. These forces are real and one cannot set  $\aleph_\mu$  or  $\mathcal{P}$  to zero without losing them. An estimate of the coordinate dependence of  $\aleph_\mu$  yields Newton’s approximation for the metric tensor. At large distances, we have  $\aleph \propto 1/r^2$ , as it follows from Eq.(4.17). A startling connection of the field  $\mathcal{P}$  with the localization of the Dirac field and origin of its mass is discussed in Sec.IV.

The physical origin of these forces can be understood from another perspective. Unlike all other terms of this equation, an additional term in Eq.(3.14) (the r.h.s. of Eq.(3.16)) accounts for the mixing of the left and right components of the Dirac field. It can be rewritten as  $m(\partial_a \mathcal{S} - D_a \mathcal{S})$ . The first term accounts only for displacement of the wave form considered as an object. The second term also accounts for the change of the internal polarization of the wave field. The difference between them is a force, which is due to internal polarization. This fact motivates the view on coordinates, as descendants of the polarization structure of the Dirac field, which was proposed in Sec.II A. Its dynamics are described by Eq.(B.10). An immediate consequence of the existence of an internal dynamic in a localized Dirac waveform is a view of pions as one of polarizations of the Dirac field (see Appendix B).

#### IV. DIRAC FIELD AND CONGRUENCES OF CURVES.

In this section, we closely follow the ideas of the intrinsic geometry of Ricci and Levi-Civita as they are presented in the monograph [10]; the metric properties of the spacetime are expressed in terms of rotations of the local coordinate pyramid. The main subject of the analysis are Eqs.(3.13) and (B.4), which are the differential identities that follow from the Dirac equations. Eq.(3.13) is trivially satisfied only for plane waves, i.e., when  $\mathcal{P} = 0$  and the tensor  $T_{ab}$  is symmetric. These solutions are employed in scattering theory and they do not represent particles. In such a context, equations like (3.13) and (B.4) cannot even be derived. Unlike the commonly known identities (3.2) and (3.3), Eqs.(3.13) and (B.4) are not covariant in the sense that they explicitly depend on congruences, which are the physical characteristics of the Dirac field. It appears that these identities impose important limitations on the properties of the metric, which is compatible with the localized solutions of the Dirac equations. These limitations are studied in the following section.

##### A. Vector current and timelike congruence.

The Ricci coefficients are real-valued and skew-symmetric in the first two indices. The tensor  $T_{ab}$  is

<sup>3</sup> These two equations could have been derived immediately in this coordinate form, in which case the analysis of anomaly in covariant derivatives of the  $T_\nu^\mu$  would have been less straightforward.

neither real nor symmetric. The r.h.s. of Eq.(3.13) is real. Therefore, the imaginary part of Eq.(3.13) is just  $\text{Im}(T_{ac} - T_{ca}) = D_c(\psi^+ \alpha_a \psi) - D_a(\psi^+ \alpha_c \psi) = \nabla_c j_a - \nabla_a j_c = 0$ , and it should be considered together with the equation  $\nabla_a j^a = 0$  of the vector current conservation. Since  $\nabla_a j_b$  are the invariants of a true tensor,  $\nabla_\mu j_\nu$ , we have two tensor equations,

$$\nabla_\mu j_\nu - \nabla_\nu j_\mu = 0. \quad (4.1)$$

and Eq.(3.2),  $\nabla_\mu j^\mu = 0$ .

When the invariant density of the Dirac (spinor) matter is positive,  $\mathcal{R} = \sqrt{j^2} > 0$ , the vector field  $j^\mu(x)$  is strictly timelike<sup>4</sup>; its tangent unit vector is  $e_{(0)}^\mu(x)$ ,  $j^\mu = \mathcal{R} e_{(0)}^\mu$ . Therefore, Eq.(4.1) becomes

$$\nabla_\mu e_\nu^{(0)} - \nabla_\nu e_\mu^{(0)} + e_\nu^{(0)} \partial_\mu \ln \mathcal{R} - e_\mu^{(0)} \partial_\nu \ln \mathcal{R} = 0. \quad (4.2)$$

Contracting this equation with  $e_{(a)}^\nu e_{(b)}^\mu$ ,  $a, b = 1, 2, 3$  and recalling Eqs.(2.19) we find that

$$\omega_{0ab} - \omega_{0ba} = 0, \quad a, b = 1, 2, 3 \quad (4.3)$$

which is a necessary and sufficient condition for the congruence  $e_{(0)}^\mu$  to be normal [9, 10]. Namely, there exists such a function,  $\mathcal{T}(x)$ , that the vector field  $e_{(0)}^\mu(x)$  is orthogonal to the family of surfaces  $\mathcal{T}(x) = \text{const}$ ,

$$\partial_\mu \mathcal{T}(x) = f(x) e_{(0)}^\mu(x), \quad (4.4)$$

where  $f(x)$  is a coordinate scalar. Contracting Eq.(4.2) with  $e_{(0)}^\nu$  we get

$$\partial_\mu \ln \mathcal{R} = e_{(0)}^\mu \partial_{(0)} \ln \mathcal{R} - \omega_{b00} e_\mu^{(b)}, \quad (4.5)$$

where  $\partial_{(0)} \ln \mathcal{R} = e_{(0)}^\mu \partial_\mu \ln \mathcal{R}$ , is the derivative in the direction of the arc  $s_0$ . Contraction of Eq.(4.2) with  $e_{(0)}^\nu e_{(a)}^\mu$  yields

$$(\partial \ln \mathcal{R} / \partial s_a) = -\omega_{a00}, \quad a = 1, 2, 3, \quad (4.6)$$

which indicates that congruences of lines, defined by the system of equations (2.14),  $dx^\mu/ds_0 = e_{(0)}^\mu$ , must experience permanent bending (acceleration) whenever the invariant density  $\mathcal{R}(x)$  of the Dirac field is not uniformly distributed. The spatial gradient of  $\mathcal{R}(x)$  cannot vanish for any localized state. Even more, the congruence of lines of the Dirac current is not a geodesic congruence, since, for geodesic lines, the vector of geodesic curvature would have vanished, i.e.,  $\omega_{a00} = 0$ .

Additional information can be extracted from Eq.(3.2). From definition (2.20) it follows that

$$\nabla_\nu e_{(0)}^\nu = -(\partial \ln \mathcal{R} / \partial s_0) = \sum_a \eta_{(a)} \omega_{0aa}. \quad (4.7)$$

Hence, we can rewrite (4.5) as

$$\partial_\mu \ln \mathcal{R} = -e_{(0)}^\mu \sum_a \eta_{(a)} \omega_{0aa} - \omega_{b00} e_\mu^{(b)}, \quad (4.8)$$

which shows that the r.h.s. of Eq.(4.8), which contains only geometric objects, is a component of a gradient. Together with condition (4.3) this constitutes a necessary and sufficient condition that the function  $\mathcal{T}(x)$ , defined by Eq.(4.4), is an harmonic function [9],

$$\square \mathcal{T} = g^{\mu\nu} \nabla_\mu \nabla_\nu \mathcal{T} = 0. \quad (4.9)$$

We may take the parameter  $t$  of  $\mathcal{T}(x) = t = \text{const}$  as a definition of the world time. For the harmonic function,  $\mathcal{T}(x)$ , the conditions of integrability for the system (4.4) of partial differential equations reads as [9]

$$\partial_\mu \ln f = -e_{(0)}^\mu \sum_a \eta_{(a)} \omega_{0aa} - \omega_{b00} e_\mu^{(b)}.$$

Comparing it with (4.8) we find that  $f(x) = \mathcal{R}$ , so that the world time  $t$  and the ‘‘proper time’’  $s_0$  are related by

$$dt = \mathcal{R} ds_0 = ds_0 / \sqrt{g_{00}}. \quad (4.10)$$

Hence, we can draw the major conclusion that: *The proper time,  $s_0$ , flows more slowly than the world time,  $t$ , whenever Dirac matter has a magnified density.* Because of the wave nature of the Dirac field, its localization becomes inevitable and as universal as free fall.

Since the congruence  $e_{(0)}^\mu$  appeared to be normal, the hypersurfaces  $\mathcal{T}(x) = t = \text{const}$  represent space at different times  $t$ . The three other vectors  $e_{(i)}^\mu(x)$ ,  $i = 1, 2, 3$  of local tetrad are spacelike, orthogonal to  $e_{(0)}^\mu(x)$  and thus belong to such hypersurfaces (by the definition,  $e_{(i)}^\mu \partial_\mu \mathcal{T} = 0$ ). The interval becomes as

$$ds^2 = g_{00} dt^2 + g_{ik} dx^i dx^k. \quad (4.11)$$

Accordingly,  $e_{(0)}^\mu = (1/\sqrt{g_{00}}, \vec{0})$ ,  $e_\mu^{(0)} = (\sqrt{g_{00}}, \vec{0})$ . If Dirac matter is in a stable configuration (with  $\mathcal{R}^2 > 0$ ), then there is a well defined time and one can consistently speak of a (quantum) state of the Dirac field<sup>5</sup>.

Because  $\mathcal{T}(x)$  is an harmonic function, it can be discontinuous on characteristic surfaces of Eq. (4.9) (the wave fronts) that separate segments of spacetime with the metric determined by the Dirac field in stable configurations, i.e., with everywhere timelike vector current. In Sec. VII these wave fronts are associated with the neutrinos.

<sup>4</sup> We always assume this. The exceptional case of lightlike congruences with  $\mathcal{R}^2 = 0$  is discussed in Sec. VII.

<sup>5</sup> In this way one can easily dismiss the paradox (attributed to R. Cutkosky), which arises in relativistic theory of bound states: hydrogen atom consisting of the yesterday’s proton and today’s electron.

Equation (3.2) of the vector current conservation now reads as

$$\partial_\mu(\sqrt{-g}e_{(0)}^\mu \mathcal{R}) = \partial_t(\sqrt{-g} g^{00}) = 0. \quad (4.12)$$

This can be recognized as the condition for the coordinate  $x^\mu$  to be harmonic,

$$\square\varphi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial\varphi}{\partial x^\mu} \right) = 0,$$

which is specified for the normal coordinate  $\varphi = \mathcal{T} = x^0$  (see, e.g. [11], §41). From (4.12), there follows one more (very intuitive) form of the current conservation,

$$\partial_t(\mathcal{R}\sqrt{-g}) = 0, \quad (4.13)$$

where  $g = \det|g_{ik}|$  is the determinant of the spatial metric. This form means that when the density  $\mathcal{R}$  grows and local time slows down, then the measure of space volume shrinks. Since the variation of  $\mathcal{R}$  and time slowdown both are intimately connected with acceleration, the last equation unites them and the Lorentz contraction of a localized object *in one physical process*. One should not even refer to a spacelike interval between two events on the opposite sides of an elementary object.

### B. Axial current and spatial part of metric.

The axial current  $\mathcal{J}^\mu$  is spacelike and orthogonal to the vector  $j^\mu$ . According to Eq.(3.3), the axial current has a source  $2m\mathcal{P}$ , which is localized together with the invariant density  $\mathcal{R}$ . Since there is no flux of vector current in this direction (the amount of matter inside a closed surface remains the same), we associate the radial direction  $e_{(3)}^\mu(x)$  with the axial current,  $\mathcal{J}^\mu = \mathcal{R}e_{(3)}^\mu$ . Then Eq.(3.3) takes form

$$\nabla_\mu e_{(3)}^\mu + e_{(3)}^\mu \partial_\mu \ln \mathcal{R} = 2m\mathcal{P}/\mathcal{R} = 2m \sin \Upsilon. \quad (4.14)$$

On the one hand, by definition,

$$\nabla_\mu e_{(3)}^\mu = \sum_a \eta_{(a)} \omega_{3aa} = \omega_{300} - \omega_{311} - \omega_{322}.$$

On the other hand, according to Eq.(4.6), we have

$$e_{(3)}^\mu \partial_\mu \ln \mathcal{R} = \partial \ln \mathcal{R} / \partial s_3 = -\omega_{300}.$$

Substituting these expressions into Eq.(4.14) we obtain an extremely important relation,

$$\omega_{131} + \omega_{232} = 2m \sin \Upsilon. \quad (4.15)$$

This can be read in different ways. First and foremost, it expresses the curvature of the two-dimensional surface  $(s_1, s_2)$  of angular coordinates via the local parameter  $\sin \Upsilon = \mathcal{P}/\mathcal{R}$  of the Dirac field. *Vice versa*, once geodesic curvatures  $\omega_{131}$  and  $\omega_{232}$  are known in advance

(e.g., from an alleged symmetry, experiment, etc.) then  $\Upsilon(x)$  is known as an explicit function of spacetime coordinates and there remains no freedom of “chiral” transformations, like in Eq.(A.8).

Second, the l.h.s. of (4.15) can be a well-defined geometric object (at least when the congruence  $e_{(3)}^\mu$  is normal and the radial coordinate is well-defined). In this case, we must have  $D_a \Upsilon = 0$  because the covariant differential operator  $D_a$  is defined only by its action on the Dirac field. Consequently, by virtue of Eq.(A.8),  $D_a \Upsilon[\psi] = \partial_a \Upsilon + 2g\aleph_a$ , we have

$$2g\aleph_a = -\partial_a \Upsilon, \quad (4.16)$$

i.e. the field  $\aleph_a$  must be a gradient. If the congruence  $e_{(3)}^\mu$  is not normal, then any symmetry of any explicit solutions of the Dirac equation with respect to arcs  $s_1$  and  $s_2$  should be considered a *dynamical internal symmetry*.

Third, for a concave surface the curvature is positive so that  $0 < \Upsilon < \pi/2$ . For the normal orthogonal spherical coordinates we have  $\omega_{131} = \omega_{232} = 1/r$  and if such a coordinate system were possible we would immediately know that

$$\begin{aligned} \Upsilon[\psi] &= \arcsin(1/mr), \quad mr > 1 \\ 2g\aleph_3[\psi] &= -\partial_r \Upsilon = \frac{1}{r\sqrt{m^2 r^2 - 1}}. \end{aligned} \quad (4.17)$$

Obviously, this simple formula cannot be exact; rather it predicts the correct asymptotic behavior at large distances.

Fourth, the condition  $|\sin \Upsilon(x)| < 1$  defines the mass parameter  $m$  as the upper limit of a possible curvature, which is, in fact, a *definition of mass from the perspective of the internal structure of a Dirac particle*. (In spherical case we would have  $mr > 1$ ; the radius must exceed the Compton length!) This result is also in agreement with the *kinematic* Lorentz contraction of special relativity. An accelerated particle is Lorentz contracted and both  $\mathcal{R}$  and the maximal curvature become  $\propto 1/\sqrt{1-v^2}$ .

In order to facilitate further analysis of the real part of Eq.(3.13), let us rewrite it's l.h.s. in terms of the axial current. Let us use the dual representation of the axial current as  $\epsilon^{stua} \mathcal{J}_a = i\psi^+ \alpha^s \rho_1 \alpha^t \rho_1 \alpha^u \psi$ , ( $s, t, u, \neq$ ), and differentiate it. The result reads as

$$D_u \epsilon^{stua} \mathcal{J}_a = i \sum_{u \neq s, t} D_u (\psi^+ \alpha^s \rho_1 \alpha^t \rho_1 \alpha^u \psi). \quad (4.18)$$

If we extend here the sum over all values of  $u$  (this sum vanishes by virtue of the equations of motion) and subtract the terms with  $u = s$  and  $u = t$ , the result will be

$$\begin{aligned} D_u \epsilon^{stua} \mathcal{J}_a &= -i\psi^+ \alpha^s \vec{D}_t \psi + i\psi^+ \alpha^t \vec{D}_s \psi \\ &\quad - i\psi^+ \overleftarrow{D}_s^+ \alpha^t \psi + i\psi^+ \overleftarrow{D}_t^+ \alpha^s \psi, \end{aligned}$$

where the r.h.s is four times the anti-symmetric Hermitian part of the energy momentum tensor. Therefore, the

real part of Eq.(3.13) reads as

$$(1/4)\omega_{acb} \cdot \epsilon^{acst} \cdot \nabla_s \mathcal{J}_t = 2mg\mathcal{P}\aleph_b, \quad (4.19)$$

and can be immediately recognized as the dual to Eq.(B.5), derived for the stress tensor,

$$(1/2)\omega_{acb}\nabla_c \mathcal{J}_a = -2gm\mathcal{S}\aleph_b. \quad (B.5)$$

These two equations clearly indicate that *any motion of the Dirac field follows the path of a helix*. The acceleration  $\omega_{0ia}$  in the direction  $s_i$  is inevitably accompanied by the spatial rotation  $\epsilon^{ijk}\omega_{jka}$  in the plane perpendicular to  $s_i$ . In plain words, *the Dirac field cannot be accelerated without causing a rotation thus behaving as a (relativistic) system of inertial navigation*.

By introducing the dual coefficients of rotation,  $\tilde{\omega}_{abc} = (1/2)\epsilon_{abst}\omega_{stc}$ , and by using Eq.(2.10), Eqs.(4.19) and (B.5) can be cast in a symmetric form,

$$\begin{aligned} (\tilde{\omega}_{abc} \cos \Upsilon + \omega_{abc} \sin \Upsilon) \nabla_a \mathcal{J}_b &= 0, \\ (\tilde{\omega}_{abc} \sin \Upsilon - \omega_{abc} \cos \Upsilon) \nabla_a \mathcal{J}_b &= 2m \cdot \mathcal{R} \cdot 2g\aleph_c, \end{aligned} \quad (4.20)$$

which shows that the phase shift  $\Upsilon$  between the left and right components of the Dirac spinor governs the balance between  $\omega_{abc}$  and  $\tilde{\omega}_{abc}$  (rotation around and acceleration along the same axis). Because the number of the observed elementary stable localized objects in Nature is very limited (proton, electron and, tentatively, neutron), this balance must be unique and very delicate. An external influence which amplifies the acceleration above certain threshold may destabilize the localized object (see further discussion in Sec. VII).

In Eqs.(4.19) and (B.5),  $\nabla_s \mathcal{J}_t = e_{(s)}^\mu e_{(t)}^\nu \nabla_\mu \mathcal{J}_\nu$ . Since  $\mathcal{J}^\mu = \mathcal{R}e_{(3)}^\mu$ , we further have

$$\begin{aligned} D_s \mathcal{J}_t &= \mathcal{R}e_{(s)}^\mu e_{(t)}^\nu \nabla_\mu e_{(s)}^{(3)} + \delta_t^3 e_{(s)}^\mu \partial_\mu \mathcal{R} \\ &= \mathcal{R}[\omega_{3ts} + \delta_t^3 (\partial \ln \mathcal{R} / \partial s_s)] . \end{aligned}$$

Finally, using Eqs.(4.6) and (4.7), which define  $\omega_{s00}$  and  $\omega_{0aa}$  as the functions of the Dirac field, we arrive at

$$\begin{aligned} (1/4) \omega_{acb} \cdot \epsilon^{acst} [\omega_{3ts} - \delta_t^3 \omega_{s00} - \delta_t^3 \delta_s^0 \omega_{0aa}] &= m \sin \Upsilon \cdot 2g\aleph_b, \\ (1/2) \omega_{stb} \cdot [\omega_{3ts} - \delta_t^3 \omega_{s00} - \delta_t^3 \delta_s^0 \omega_{0aa}] &= -m \cos \Upsilon \cdot 2g\aleph_b, \end{aligned} \quad (4.21)$$

At this point, we can conclude that the parameters  $g\aleph_b$  in the connection  $\Gamma_b$  (A.3) are totally defined by the bending of the system of congruences. Despite that fact that the Lagrangian for the Dirac equations (3.1) includes the term  $\mathcal{J}^a \aleph_a$ , which can be interpreted as an interaction between the axial current and the field  $\aleph_a$ , it cannot be viewed as an independent field that is governed by an additional equation of motion. (Otherwise, such equations must be *invented*, which we, so far, tried to avoid.)

### C. The case of the normal radial coordinate. Qualitative consequences of the localization.

Even for the localized waveforms, the existence of the surface of a constant distance from a center is not given *gratis*. Such a surface must be orthogonal, at every point, to the tangent vectors  $e_{(3)}^\mu$  of the congruence of lines of the axial current. Unlike the previously studied case of the timelike congruences  $e_{(0)}^\mu$ , the corresponding conditions for integrability do not universally follow from the equations of motion. Most likely, the radial coordinate cannot be normal. However, sometimes (mostly for long-lived particles) empirical data may hint that such a normal hypersurface of a constant distance  $s_3$  from a center, which is spanned by the “angular” arcs ( $s_1, s_2$ ) with tangent unit vectors (2.7) may be a good approximation.

This is what we intuitively expect in a one-body problem and we have to verify that this assumption is consistent with the equations of motion and the established earlier constraints. In what follows, we consider Eqs.(4.17) as the criterion of the spherical symmetry and try to profit from the fact that Eqs.(4.21) significantly simplify under the assumption that the congruence  $e_{(3)}^\mu$  is a normal congruence (which, possibly, can be a first approximation in a sequence of iterations). This will also allow us to qualitatively understand the trends in critical behavior of the tetrad vectors near the limit surface,  $\sin \Upsilon \rightarrow 1$ , and re-discover some well known properties of matter (which cannot be done in a picture of matter as plane waves).

Since the congruence  $e_{(3)}^\mu$  is set normal, there should exist a function  $\mathcal{N}(x)$  such that

$$\partial_\mu \mathcal{N}(x) = \zeta(x) e_{(3)}^\mu(x), \quad (4.22)$$

where  $\zeta(x)$  is a coordinate scalar. The hypersurfaces  $\mathcal{N}(x) = r = \text{const}$  are the surfaces of radius  $r$ , i.e.,

$$dr = \zeta ds_3 = ds_3 / \sqrt{g_{33}}. \quad (4.23)$$

From the integrability condition for Eq.(4.22) it is straightforward to derive the equations (which are similar to equations for the the function  $f(x)$  (cf.(4.4))

$$-\partial_\mu \ln \zeta = e_{(3)}^\mu \partial_{(3)} \ln \zeta - \omega_{a33} e_{(3)}^{(a)}, \quad (4.24)$$

$$(\partial \ln \zeta / \partial s_a) = \omega_{a33} , \quad (a = 0, 1, 2); \quad (4.25)$$

but, we have no constraint that would express  $\zeta$  as a function of the Dirac field. For the normal congruence  $e_{(3)}^\mu$  we have  $\omega_{3ab} = \omega_{3ba}$  for  $a, b \neq 3$ ,  $a \neq b$ , as a necessary and sufficient condition [9, 10] and, consequently, the first term in brackets in Eqs.(4.21) simplifies,  $\omega_{stb}\omega_{3st} = -\omega_{03b}\omega_{033} + \omega_{13b}\omega_{133} + \omega_{23b}\omega_{233}$  and  $\omega_{acb}\epsilon^{acst}\omega_{3st} = \omega_{12b}\omega_{033} + \omega_{02b}\omega_{133} + \omega_{01b}\omega_{233}$ . In the second term, the sum includes only  $a, c = 1, 2$ . Because we assume a stable object, the third term in brackets cancels,  $\partial \ln(\mathcal{R}\zeta)/\partial s_0 \approx 0$ . Hence, the system of Eqs.(4.21) can be cast as

$$\begin{aligned} (\omega_1\omega_{02b} - \omega_2\omega_{01b}) &= 2mg\aleph_b \sin \Upsilon, \\ (\omega_1\omega_{31b} + \omega_2\omega_{32b}) &= 2mg\aleph_b \cos \Upsilon, \end{aligned} \quad (4.26)$$

where

$$\omega_j = \frac{\omega_{j00} + \omega_{j33}}{2} = \frac{1}{2} \frac{\partial \ln(\zeta/\mathcal{R})}{\partial s_j}, \quad j = 1, 2.$$

Then, Eqs.(4.26) with  $b = 0, 1, 2$  (and  $\aleph_0 = \aleph_1 = \aleph_2 = 0$ ) yield a set of six equations,

$$\frac{\omega_1}{\omega_2} = -\frac{\omega_{320}}{\omega_{310}} = \frac{\omega_{011}}{\omega_{021}} = \frac{\omega_{012}}{\omega_{022}} = \frac{\omega_{321}}{\omega_{311}} = \frac{\omega_{232}}{\omega_{312}} = \pm 1, \quad (4.27)$$

where the rightmost equation immediately follows from  $\omega_{312}^2 = \omega_{131}\omega_{232} = 1/r^2$  and  $\omega_{131} = \omega_{232} = 1/r$ . In spherical case,  $\aleph_3$  is given by Eqs.(4.17) and, consequently,

$$\begin{aligned} 2\omega_1\omega_{023} &= \frac{1}{r^2\sqrt{m^2r^2 - 1}}, \quad \omega_1(\omega_{313} \pm \omega_{323}) = \frac{1}{r^2}, \\ \omega_2 &= \pm\omega_1, \quad \omega_{013} = \mp\omega_{023}, \quad \omega_{312} = \omega_{321} = \pm 1/r. \end{aligned} \quad (4.28)$$

As one could expect, in the case of normal radial congruences, there is a full symmetry between congruences of arcs  $ds_1$  and  $ds_2$ . At large distances we generally have  $mr \gg 1$  so that spatial rotations dominate. *Vice versa*, near the inner boundary  $mr = 1$  the accelerations  $\omega_{013}ds_3$  and  $\omega_{023}ds_3$  in tangent directions (as well as accelerations  $\omega_{031}ds_1$  and  $\omega_{032}ds_2$  in radial directions) become infinite. When, starting from a generic point  $x_0$ , we move along lines of congruences  $e_{(i)}^\mu(x)$  approaching  $r \sim m^{-1}$ , then local tetrad rotate (with respect to the  $e_{(i)}^\mu(x_0)$ ) in such a way that *all* directions become nearly lightlike, so that the tangent velocities  $v_i = \dot{s}_i \rightarrow c$ . These observations explain the formally derived inner boundary  $r = 1/m$  (generally,  $|\sin \Upsilon| = 1$ ) of the Dirac particle as the caustic of the lines of the Dirac currents.

In fact, we have two interconnected mechanisms of the time slowdown (due to the amplified  $\mathcal{R}$  and because the vector currents tend to approach the lightlike directions), which cannot be separated. From the perspective of an “external observer”, the time flow literally stops at the critical surface of a stable Dirac waveform. Therefore, the

sharp interaction of the deeply inelastic scattering always resolves an apparently static object in a random configuration determined by the foregoing causal evolution (cf. discussion of evolution equations of QCD in Ref.[12]). Certain patterns of symmetry observed in such processes most likely correspond to the symmetry of (the critical points of) projection of the actual currents onto a surface determined by the collision axis, the collision plane, etc. These patterns well may have very little to do with the internal dynamics of the stable waveform.

When  $mr \gg 1$ , we generally have  $\sin \Upsilon \rightarrow 0$  and, according to Eq.(2.11), the pseudoscalar density nearly vanishes and  $\mathcal{S} \approx \mathcal{R}$ . Magnetic polarization of the Dirac field is greater than the electric one,  $|\vec{L}| \gtrsim |\vec{K}|$ . *Vice versa*, at the shortest possible distances, when  $|\Upsilon| \rightarrow \pi/2$ , the Lorentz boosts play a major role. Accordingly, the pseudoscalar density  $\mathcal{P} \approx \mathcal{R}$  is large and electric polarization is dominant,  $\vec{K}^2 - \vec{L}^2 \approx \mathcal{R}^2 \gg 1$ . At large distances an appropriate choice for  $e_{(1)}^\mu$  and  $e_{(2)}^\mu$  will be vectors  $H_i$  and  $\tilde{H}_i$  of Eqs.(2.7). Close to the critical surface, where the mass of the particle is being formed, these will be  $E_i$  and  $\tilde{E}_i$ .

The impossibility to introduce a normal orthogonal coordinate system in the presence of the axial potential  $\aleph$  in the equations of motion is explicitly illustrated in Appendix C. An attempt to separate the angular variables in the Dirac equation in the presence of only the radial component  $\aleph_r(r)$  is made. In this case, the radial coordinate,  $r$ , is a well defined normal coordinate. It appears that even in such a simplest case there are no operators with eigenvalues of angular momentum that commute with the Hamiltonian. At the same time, angular variables can be explicitly separated in the equations of motion. The only possible explanation of this fact is that the formally introduced angles do not represent arcs of the usual spatial angular coordinates. There is no reason to require that solutions of the Dirac equations must be single-valued along these arcs. However, equations (C.8) for angular functions are clearly the equations for spherical harmonics. These  $SU(2)$  harmonics can be interpreted only as elements of an internal (dynamical) symmetry in the space of polarizations of the Dirac field. In order to find out what can be the non-geometric integrals of motion the angular and radial functions must be studied together.

## V. NONLINEAR DIRAC EQUATION.

In this section, following the programme outlined in Sec.II B, we will incorporate the nonlinear effects, which so far were found as constraints, into the Dirac equation. Following Fock [3], let us rewrite the operator  $\alpha^a\Omega_a$  as

$$\alpha^b\Omega_b = (1/2)\omega_{aca}\alpha^c - (i/4)\epsilon_{acbd}\omega_{acb}\rho_3\alpha^d. \quad (5.1)$$

Then, the Dirac equation reads as

$$\alpha^b \left[ \frac{\partial}{\partial s_b} + ieA_b - \frac{1}{2}\omega_{aba} + i\rho_3(g\aleph_b + \frac{1}{2}\dot{\omega}_{aba}) \right] \psi = -im\rho_1\psi, \quad (5.2)$$

where  $\aleph_b$  and  $\mathbf{w}_b = -(1/2)\epsilon_{acdb}\omega_{acd} = -\dot{\omega}_{aba}$  are the sets of invariants; the latter differ from zero whenever spacetime does not admit a coordinate net of all normal congruences. In general, the invariants  $\mathbf{w}_b$  do not vanish and they are complementary to the invariants  $\aleph_b$  given by Eqs. (4.20) and (4.21).

According to Eqs.(4.6) and (4.7), each sum  $\sum_a \eta_{(a)}\omega_{aba}$  includes either the terms  $\omega_{0i0} = \partial \ln \mathcal{R}/\partial s_i$  ( $i = 1, 2, 3$ ) or  $\sum_i \omega_{i0i} = \partial \ln \mathcal{R}/\partial s_0$ , all of which are

bilinear forms of  $\psi^+$  and  $\psi$  and, geometrically, are the geodesic curvatures. At first glance, the presence of these curvatures makes the Dirac equation extremely non-linear. This genuine nonlinearity, however, can be effectively alleviated after the following “normalization” of the Dirac wave function. If we observe that

$$\frac{\partial \psi}{\partial s_a} - \frac{1}{2} \frac{\partial \ln \mathcal{R}}{\partial s_a} \psi = \sqrt{\mathcal{R}} \frac{\partial}{\partial s_a} \left( \frac{\psi}{\sqrt{\mathcal{R}}} \right) \quad (5.3)$$

and assume Eq.(4.16) be true, then we arrive at a much simpler equation for the normalized function  $\xi = \psi/\sqrt{\mathcal{R}} = (g_{00}[\psi])^{1/4}\psi$ :

$$\left[ i \frac{\partial}{\partial s_0} - eA_0 - \frac{1}{2}\rho_3[\mathbf{w}_0 + \partial_0 \Upsilon] - \sum_{i=1}^3 \alpha^i \left( i \frac{\partial}{\partial s_i} + i \frac{k_i}{2} - eA_i + \frac{1}{2}\rho_3[\mathbf{w}_i + \partial_i \Upsilon] \right) - m\rho_1 \right] \xi = 0, \quad (5.4)$$

where  $k_i = \sum_{j \neq i} \omega_{jij}$ . The nonlinear Dirac equation (5.2) now looks like a *linear equation* for the normalized function  $\xi$ . Once  $m^{-1}$  is accepted as a measure of length, this equation is dimensionless and does not change under a similarity transformation. At the first glance, the term  $\rho_3 \partial_i \Upsilon$  in it is always nonlinear; however, the constraint  $k_3 = 2m \sin \Upsilon = 2m(\xi^+ \rho_2 \xi)$  (cf. Eq.(4.15)) eliminates the non-linearity whenever the curvature  $k_3$  can be determined as a function of coordinates from geometric considerations. In the one body problem this curvature always has the meaning of the inverse radius of the enveloping convex surface. At least at large distances (at the scale of  $1/m$ ) we have  $\partial_r \Upsilon \propto -1/mr^2$ , which brings in a singular potential  $\propto 1/r^2$  into the Dirac equation and a Newton’s force into Eqs.(3.13)-(3.16). Such a construct may serve as the first step of an iterative procedure for the Dirac equation.

In general, there is no direct connection between the radius of curvature and the distance to any distinct point inside a localized object; the gradients  $\partial_i \Upsilon$  can be arbitrary large. Most likely, the solutions of Eq. (5.2) have multiple caustics where the invariant density  $\mathcal{R}$  is large and  $\mathcal{P}$  dominates to the extent that  $\mathcal{R} \approx \mathcal{P}$ . So far, we did not find in mathematical literature regular methods to study equations like (5.2). The methods of contact geometry [13] seem to be most relevant.

The most important source of the nonlinearity of the Dirac equation resides in that fact that evolution, in terms of proper time  $s_0$ , has a different rate at different points of the localized object; the Lorentz invariance is explicitly broken in its interior. The rate of evolution,  $\partial/\partial s_0$ , along *this* time cannot yield anything like the energy of this object as a whole. Fortunately, we have proved that a stable object does have a well defined hypersurface of a constant world time  $t$ . Therefore, a meaningful evolution scale for a stable object as a whole is associated with the macroscopic time  $t$ . According to Eq.(4.10), we have  $dt = \mathcal{R} ds_0$ . Hence,

$$\left[ i\mathcal{R} \frac{\partial}{\partial t} - eA_0 - \frac{1}{2}\rho_3[\mathbf{w}_0 + \partial_0 \Upsilon] - \sum_{i=1}^3 \alpha^i \left( i \frac{\partial}{\partial s_i} + i \frac{k_i}{2} - eA_i + \frac{1}{2}\rho_3[\mathbf{w}_i + \partial_i \Upsilon] \right) - m\rho_1 \right] \frac{\psi}{\sqrt{\mathcal{R}}} = 0, \quad (5.5)$$

and now this equation has a scale fixing factor  $\mathcal{R}$  in front of the  $\partial/\partial t$ . This effect is a major one because it corresponds to a clearly understood physical effect, refraction of the Dirac waves. It is of the same physical origin as self-focusing in nonlinear optics and acoustics or deflection of light in the gravitational field of a star. In fact,  $\mathcal{R}$  plays a role of “refractive index” depending on the amplitude. Since the phase velocity decreases with increasing amplitude, the field tends to auto-localize. This mechanism of concentration is the most distinctive property of gravity which may signal its role in matter formation

from fields at all scales. The spatially uniform solutions of the Dirac equation just cannot be stable.

In Majorana representation of the Dirac matrices all the matrices  $\alpha^a$ ,  $i\rho_1$  and  $i\alpha^a\rho_3$  become real. The complex conjugation of  $\psi$  still amounts to changing of the signs of the coupling constant  $e$  and of  $\partial/\partial s_0$ . But now the latter does not represent energy. It must be replaced by  $\mathcal{R}\partial/\partial t$ , so that the charge conjugation in Eq. (5.5) is not just a discrete mathematical transformation – the density  $\mathcal{R}(x)$  is different for the positive and negative charges.

## VI. INTERACTION OF LOCALIZED OBJECTS: ELECTRIC CHARGE, CP-SYMMETRY, METRIC, RADIATION

Our major perception regarding the vacuum is the absence of matter. Since matter inevitably is localized, this means that in the vacuum,  $\mathcal{R}$  is constant and the space-time metric of the Lorentz vacuum has  $g_{00} = c^2 = 1$ . At the same time, we have Eq.(4.10),

$$dt = \mathcal{R}ds_0 = ds_0/\sqrt{g_{00}}. \quad (4.10)$$

Therefore, the empirical  $g_{00} = 1$  of an empty space corresponds to  $\mathcal{R} = 1$ . This state cannot be stable. Due to a very special nonlinearity of Eq.(5.5) Dirac waves tend to refract towards domains where  $\mathcal{R} - 1 > 0$  amplifying  $\mathcal{R}$  there until some saturation level (or caustic) is reached and an external boundary is formed. The opposite trend must be observed in domains where  $\mathcal{R} - 1 < 0$ ; the Dirac waves tend to escape them. This conjecture can be phrased more precisely as:

*Identification of the sign of  $(\mathcal{R} - 1)$  with the sign of electric charge leads to a dynamic picture of an empirically known charge-asymmetric world in which stable positively charged elementary Dirac objects are highly localized (and presumably heavy) while negatively charged objects tend to be poorly localized (and presumably light).*

The best prospect of this idea is that these objects are the protons (or nuclei) and the electrons of the real world. When electric forces come into play, the electrons become somewhat localized around heavy objects, thus forming electrically neutral matter. For atomic electrons, the effect of  $|\mathcal{R} - 1| \ll 1$  on the metric must be negligible; they are smeared over distances much exceeding the Compton length and held near nuclei by electric forces. The view of a vacuum as a classical Dirac field with the standard level  $\mathcal{R} = 1$  and propagating in it waveforms, instead of an ensemble of quantum oscillators with an unbound spectrum (which are excited as plane waves and interact via their mutual scattering) has an important implication. It preserves the physical meaning of the time variable as a parameter along the lines of the vector current even in the absence of sensible matter.

The next important question is the interaction between localized objects and their transformations. Below, we show that basic relations derived in the previous sections allow one to draw quite precise conclusions/postdictions regarding the properties of different interactions in realistic matter. In the present context, these conclusions still are semi-qualitative and rely on very crude approximations; however, it is vital that all are derived from one common principle.

### A. CP-symmetry and CP-violation.

The difference in degree of localization obviously makes the localized charges of opposite sign unequivocally different particles. The conjectured correlation between the

signs of electric charge and of  $\ln \mathcal{R}$  also qualitatively explains the interdependence between the discrete  $C$ - and  $P$ -transformations as a natural property of the simplest localized waveforms. Indeed, by virtue of Eq.(4.15) (for an alleged simple symmetry), the sign of pseudoscalar density  $\mathcal{P} = \mathcal{R} \sin \Upsilon$  is in one-to-one correlation with the sign of curvature of the 2-d enveloping surface of a stable object. In vacuum,  $\ln \mathcal{R} = 0$ , there is nothing to envelope. For the positive charges we have  $\ln \mathcal{R} > 0$  and this surface is convex (positive curvature), which means that it's normal vector is directed towards the lower invariant density  $\mathcal{R}$  or *outward*. The negative charges have  $\ln \mathcal{R} < 0$ , the surface is concave and the normal vector is directed *inward*. Accordingly, the sign of curvature and of pseudoscalar density is negative. Therefore, while  $C$  qualitatively stands for the charge conjugation,  $P$  is not an abstract reflection symmetry in a flat space; it stands for the interchange of *inward* and *outward*. In a sense, these two discrete transformations do not exist separately; thus understood  $CP$ , as a physical symmetry between the corresponding processes, must be broken by nonlinear effects of the local time slowdown and self-localization. Once the metric follows matter, the  $P$ - and  $T$ -reflections cannot be considered as the purely geometric operations; moreover, there are two physically different times, local time  $s_0(x)$  and world time  $t$ . One should not forget that the existing proof of the  $CPT$ -theorem relies heavily on the Poincaré invariance, which is not compatible with the local time slowdown<sup>6</sup>.

Our major conclusion about the nature of localization is drawn from the analysis of the elementary stable waveforms. However, it relies on basic properties of the wave propagation so that it seems reasonable to apply them to at least the long-lived waveforms. From this perspective, any positively charged particle should have a slightly longer lifetime and be more localized than its negative counterpart. While the proton is small and stable, the antiproton should not have an as well defined outward boundary as the proton has. The lifetime of the anti-hydrogen might not be long even when it is completely isolated from normal matter. For neutral unstable particles like kaons the notion of  $CP$ -symmetry is even more ambiguous. Most likely, these are the waveforms without a stable shape where the curvatures  $\omega_{131}$  and  $\omega_{232}$  in Eq.(4.15) are different; they may change their sign along the surface  $(s_1, s_2)$  and even be time-dependent.

Observation of the  $CP$ -violating asymmetry in the  $K_L^0$  decays was originally considered (by L.B Okun) as an evidence that the microscopic world has its own intrinsic time arrow and an absolute definition of helicity [14]. This viewpoint, clearly supported by present work, is reiterated in an extensive review on  $CP$ -violation and

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<sup>6</sup> This is in contrast with the view of Dirac field as the representation of the Lorentz group. In that framework, the Poincaré invariance is presumed and all states can be obtained from a single state by a sequence of the Lorentz transformations.

matter-antimatter oscillations by I. Bigi [15].

Having no adequate solutions of Eq. (5.2), let us try to motivate the preferable decay of the neutral kaon  $K_L^0$  into more localized  $e^+$  or  $\mu^+$  relying on quark model, according to which  $K^0 = (d\bar{s})$  and  $\bar{K}^0 = (\bar{d}s)$ . The  $K^0$ , viewed as a bound state of two quarks, contains the heavy highly localized positively charged  $\bar{s}$  and light weakly localized  $d$ ; one may tentatively think of a confining potential. The design of  $\bar{K}^0$  is very similar; the difference is that the heavy  $s$ -quark is negative and is not localized as well as  $\bar{s}$  while light positive  $d$  is localized better than  $\bar{d}$ . By the same argument as previously, the time in  $K^0$  configuration flows more slowly and it is a more compact object than  $\bar{K}^0$ . Therefore, if we try to treat  $K^0$  and  $\bar{K}^0$  as two different quantum states, they will be *a priori* not degenerate states. Their superpositions will immediately exhibit temporal  $K^0 \leftrightarrow \bar{K}^0$  oscillations. These oscillations must be asymmetric – in terms of the world time, the phase of  $K^0$  should last a bit longer than of  $\bar{K}^0$  (and it really does! <sup>7</sup>). The decays  $K^0 \rightarrow l^+\nu\pi^-$  will be observed more frequently. This effect is implicitly encoded in the empirical parameterization [14],

$$K_L^0 = \frac{1}{\sqrt{1+\epsilon^2}} \left[ \frac{1+\epsilon}{\sqrt{2}} K^0 + \frac{1-\epsilon}{\sqrt{2}} \bar{K}^0 \right],$$

where the weight of the process  $K^0 \rightarrow l^+\nu\pi^-$  superposition is larger than of  $\bar{K}^0 \rightarrow l^-\bar{\nu}\pi^+$ .

Of course, an *ad hoc* superweak interaction or the mixing phase of the CKM matrix do provide an adequate parameterization of the existing data (see, e.g., the reviews by D.Kirkby and Y.Nir and by L. Wolfenstein *et al* in Ref.[17]) <sup>8</sup>. The advocated dynamical picture can be a viable explanation of this phenomenology. Our analysis deduces the observed in Nature  $CP$ -asymmetry as a consequence of the more fundamental charge asymmetry; unlike it was envisioned by A. Sakharov [18] in the hot matter scenario, the former is rather a complement than a prerequisite for the latter.

Transient processes explicitly violate those symmetries that are apparently seen when the long lived waveforms are considered as stable ones. Experimental discovery of such process-dependent asymmetries could have been an indication that these are the dynamical symmetries of special solutions. For example, the parameter,  $A_L(l) = [\Gamma(l^+\nu_l\pi^-) - \Gamma(l^-\bar{\nu}_l\pi^+)]/[\text{sum}]$ , of the  $K_{l3}^0$  decay can be different for the electron and muon modes. Indeed, the formation times of the  $e\nu_e$  and  $\mu\nu_\mu$  cannot be equal and one can expect that  $A_L(e) \neq A_L(\mu)$ .

<sup>7</sup> The difference  $A_T = [\text{rate}(\bar{K}^0 \rightarrow K^0) - \text{rate}(K^0 \rightarrow \bar{K}^0)]/[\text{sum}] \simeq 4\epsilon = 6.6 \cdot 10^{-3} > 0$  was measured by the CPLEAR collaboration [16]. It is a test for T-violation. Unlike conjectured in Ref. [15], it should not be attributed to the asymmetry in initial conditions.

<sup>8</sup> In the future quest for the explicit solutions of the nonlinear Dirac equation one could profit from an analytic parameterization of the entries of CKM matrix as an effective representation of the data. (See, e.g., Ref. [27])

## B. Interaction of the neutral objects.

Let us apply Eq.(3.13) [or Eq.(3.18), in the coordinate form] to the description of a Dirac waveform which consists of two neutral parts. Let one part be a heavy spherically symmetric object, which is considered, in the sense of Eq.(4.17), as the source of the radial field  $\aleph_r$ . The second part, a segment of thin shell, which is characterized by the energy-momentum  $T^\sigma_\mu$  and the pseudoscalar density  $\mathcal{P}$ , plays the role of a test particle for the metric, which is equivalent to the “external field”  $\aleph_r$ . With respect to this shell, the direction of the  $\aleph_r$  is the inward direction. Following the logic explained in Sec.III, let us assume that the radial coordinate is normal and that the energy density  $T^0_0$  is the largest component of the energy-momentum. Then Eq.(3.13) can be simplified to  $\omega_{r00} \cdot T_{00} = +2g\aleph_r \cdot m\mathcal{P}$ . Substitute here  $\omega_{r00}$  from Eq.(4.5) and  $2g\aleph_r$  from Eq.(4.17) at  $mr \gg 1$ ,  $2g\aleph_r = C/r^2$ . The constant  $C$  accounts, in a crude manner, for the unknown but potentially calculable detail structure of the central source. We arrive at

$$\partial_r \ln \mathcal{R} T_{00} = -Cm\mathcal{P}/r^2, \quad (6.1)$$

where we may roughly put  $T_{00} \approx m\mathcal{R}$  and replace  $\mathcal{P} = \mathcal{R} \sin \Upsilon$ . This leads to the equation,

$$\partial_r \ln \mathcal{R} = -C \sin \Upsilon / r^2. \quad (6.2)$$

With the boundary condition,  $\mathcal{R} \rightarrow 1$  when  $r \rightarrow \infty$ , it has an obvious solution,

$$\mathcal{R} = \exp[\kappa/r]. \quad (6.3)$$

According to Eq.(4.10), the Dirac density defines the  $g_{00}$  component of the metric Eq.(4.12) as  $g_{00} = 1/\mathcal{R}^2$ . Therefore,

$$g_{00} = \exp[-2\kappa/r] \approx 1 - 2\kappa/r, \quad (6.4)$$

which corresponds to the Newton approximation of the GR.

From the analysis of Sec.IV it is evident that stationary configurations of the Dirac field cannot be exactly static (as, e.g., cannot be static in GR a two-body gravitating system); some residual time dependence should be kept in mind. The exact equations (4.21) and their approximate solution for the case of normal radial coordinate are bridged by the approximate condition,  $\partial \ln(\mathcal{R}\zeta)/\partial s_0 \approx 0$ , of Sec.IV C. It can be rewritten as  $\partial(g_{00}g_{rr})/\partial s_0 \approx 0$ , and used as the criterion of a *nearly static* metric. Relying on this criterion, we can make a step further and determine the radial component of the metric,

$$g_{rr} = \exp[+2\kappa/r] \approx 1 + 2\kappa/r, \quad (6.5)$$

thus recovering the first post-Newton approximation of the GR. We can identify  $\kappa = C \sin \Upsilon$  with the gravitational constant,  $G$ , times the mass of the heavy object. The smallness of  $G$  corresponds to the smallness of the



phase shift  $\Upsilon$  between right and left spinor components of the interacting waveforms at a large spatial separation, which seems to be a prerequisite for their individual macroscopic stability. The above results were obtained on very different from the standard GR premises. The Einstein equations for the metric field describe the motion of *macroscopic objects*. Thus, we can reiterate our previous conjecture that they also can be a prerequisite for the stability of these objects that follows from the Dirac equation. Most likely this condition will be found in the Einstein-Infeld-Hoffmann form,  $R_{\mu\nu} = 0$ . Then at least the divergence of the imaginary part of the tensor  $T^\mu_\nu$  becomes zero.

### C. Interaction of the electric charges.

Empirically, the field  $A_c$  in the connection (A.3) is the electromagnetic field, which is responsible for the Lorentz force. The system of invariants  $\aleph_a$  in  $\Gamma_a$  is determined either by geometric properties of congruences like (4.15) or by nonlinear constraints (4.20) and (4.21). As long as the field  $\aleph_a$  is a gradient, the r.h.s. of Eq.(3.9) is a system of invariants, which include, except for the geometric terms, the term  $ej^a F_{ab}$ . In the coordinate representation (3.14), this part is translated into  $ej^\mu F_{\mu\nu}$ , and this is the only term on the r.h.s. of the *real part* of Eq.(3.9). This term is known as the Lorentz force density of the field  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ ; this is the only reason to identify  $A^\mu$  with the electromagnetic field. The vector field  $A_\mu = e_\mu^{(a)} A_a$  originates from the connection  $\Gamma_a$  and thus is an external field.

For the real part of the energy-momentum tensor,  $T_{ab} = (i/2)[\psi^+ \alpha^a \vec{D}_b \psi - \psi^+ \vec{D}_b^+ \alpha^a \psi]$ , and in the *artificial normal coordinates*, discussed in Sec.III, we had equation (3.17),

$$\frac{\partial}{\partial x^\mu} [\sqrt{-g} T^\mu_\nu] = e \sqrt{-g} j^\mu F_{\mu\nu} . \quad (6.6)$$

Due to the finite size of the waveform (or due to the normalization of the corresponding quantum state), after integration of Eq.(6.6) over the space volume, the constant  $e$  becomes the electric charge of a *particle*. The correspondence principle works only because the Dirac field waveforms are localized! As it should be, the kinematic acceleration  $w^\mu = e_{(0)}^\nu \nabla_\nu e_{(0)}^\mu$  of a charged particle does not include a gravitational part (see [11], §63).

The field  $F_{\mu\nu}$  is a tensor and it satisfies the identity (the first couple of Maxwell equations),

$$F_{\mu\sigma\nu} \equiv \nabla_\mu F_{\sigma\nu} + \nabla_\sigma F_{\nu\mu} + \nabla_\nu F_{\mu\sigma} = 0. \quad (6.7)$$

The divergence of this tensor,  $\nabla^\mu F_{\mu\sigma\nu} = 0$ , can be cast in the following form,

$$\begin{aligned} -\square F_{\mu\nu} + R^\kappa_\mu F_{\kappa\nu} - R^\kappa_\nu F_{\kappa\mu} - R_{\mu\nu\kappa\sigma} F^{\kappa\sigma} \\ = (\nabla_\mu [\nabla^\sigma F_{\sigma\nu}] - \nabla_\nu [\nabla^\sigma F_{\sigma\mu}]) , \end{aligned} \quad (6.8)$$

where  $\nabla^\sigma = g^{\sigma\lambda} \nabla_\lambda$ . The l.h.s. of this equation is the wave operator for the field  $F_{\mu\nu}$ . The two terms in the r.h.s. can be transformed further after we postulate the second couple of Maxwell equations,

$$\nabla_\mu F^{\mu\nu} = e J^\nu , \quad \nabla_\mu J^\mu = 0, \quad (6.9)$$

with the Dirac's vector current  $eJ^\mu$  in the r.h.s. This amounts to the second (in fact, independent) definition of electric charge as the divergence of the electric field, and a few reservations must be made. First, without a good reason, the same coupling constant, as in the connection  $\Gamma_a$ , is postulated. Only gauge invariance as an independent principle can provide for an unquestionable equality of these constants. Second, the Dirac field in Eq.(6.8) is assumed to be a stable configuration and the field  $\aleph_\mu$  is considered a gradient. Otherwise, the conservation of the vector current and the non-conservation of the axial current will conflict with Maxwell equations. Third, the interactions between the electromagnetic field and the spacelike axial current or pseudoscalar density, which are present in Eq.(B.10) and affect the balance of momenta inside the Dirac object, are disregarded. Only those interactions that are responsible for the change of timelike components of the momentum of a stable particle are allowed to be sources of electromagnetic field. Then Eq.(6.7) becomes,

$$\begin{aligned} -\square F_{\mu\nu} + R^\kappa_\mu F_{\kappa\nu} - R^\kappa_\nu F_{\kappa\mu} - R_{\mu\nu\kappa\sigma} F^{\kappa\sigma} &= e Q_{\mu\nu} , \\ Q_{\mu\nu} &= (\nabla_\mu J_\nu - \nabla_\nu J_\mu), \end{aligned} \quad (6.10)$$

where  $Q_{\mu\nu}$  is a convenient intermediate notation.

The crucial question is, if  $J_\mu$  in this equation is (or can be) the current  $j^\mu$  of Eq.(6.6), which was derived as a consequence of the Dirac equation with the potential  $A_c$  in the connection  $\Gamma_c$ . Evidently, the answer is *no* because then, according to Eq.(4.1), we must have  $Q_{\mu\nu} = 0$ . Therefore, an object with the well-defined proper time across its volume cannot be a source of an electromagnetic field that may result in the Lorentz force of self-interaction. The definition (6.9) *must* be complemented by Eq.(6.6), which then determines the measured acceleration of another charge that senses the field of the first one. The potential  $A_c$  in the Dirac equation that describes a localized object must be “external”; its source can be only the current of another object. The problems of mass and charge (including the problem of electromagnetic mass) are not a one-body problem. One further implication of this observation is that if one can simultaneously identify two localized objects, then the Dirac fields of these objects cannot overlap in space-time<sup>9</sup>. Since, for a stable object,

<sup>9</sup> From perspective of the second quantization, when waveforms of the Dirac field are associated with different states, this means that the Fock operators of these two states must anti-commute! With an adequate definition of the statistical ensemble, this

one cannot set  $e j^\nu = \nabla_\mu F^{\mu\nu}$  in the expression for the Lorentz force, it is also impossible to express this force as the divergence of the energy-momentum tensor, i.e., as  $e j^\mu F_{\mu\nu} = \nabla_\sigma F^{\sigma\mu} F_{\mu\nu} = -\nabla_\mu \Theta_\nu^\mu(F)$  and claim that  $\Theta_0^0$  is the energy density of the electromagnetic field. This is not surprising since one cannot convert this energy into any other form without a second object.

For an isolated stable charged object the condition that it cannot interact with its own electric field means that the only “potential” in the wave equation (5.2) is  $\aleph \propto 1/r^2$ . The wave equation with such a steep potential may have a strongly localized solution regardless of the sign of this potential. The difference from the commonly studied cases is that now the signs of this potential for the left- and right-handed components are opposite and that, for a stable wave form, the region  $|\sin \Upsilon(x)| > 1$  (e.g.,  $mr < 1$ ) is cut off by Eq.(4.15). Therefore, a precursor of a localized state is present in the Dirac equation even before the universal nonlinear mechanism of the time slowdown takes over.

The field which is measured via the Lorentz force (6.6) always is a “field in vacuum”. The wave equation (6.10) for the  $F_{\mu\nu}$  from Eq.(6.6) is a homogeneous equation, which depends on the Dirac field of (6.6) only parametrically, via derivatives of  $g_{\mu\nu}(\psi)$  in the Riemann tensor. The electromagnetic sector of the theory turns out to be entirely in the form required by Riemannian geometry. This sector is responsible for the propagation of signals that are used to synchronize macroscopic clocks (which is unambiguous only in special relativity). A stable waveform of the Dirac field with  $Q_{\mu\nu} = 0$  neither interacts with its own electromagnetic field nor can it emit an electromagnetic field, as a signal, by itself. This is yet further evidence that the object is in a stationary state.<sup>10</sup> This property is in line with the well known fact that equation of the Coulomb’s law is a constraint and not an equation of motion. The longitudinal part of the electromagnetic field does not propagate; the Coulomb field is simultaneous with its source. The field of radiation emerges only when this simultaneity is lost. This is yet another view of the realm of the well-known phenomena of transient processes where the proper field of a particle is truncated[19].

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seems to be sufficient to establish the standard connection with the statistics. These anti-commutation relations, however, cannot be immediately translated into the commutators between the coordinate-dependent operators of the Dirac field. The Fock operators belong to the linear Hilbert space of the quantum states, while Dirac equation that yields these states is nonlinear.

<sup>10</sup> It is important to emphasize that the l.h.s. of Eq.(6.10) is given in terms of measurable electric and magnetic fields; therefore we indeed are dealing with a signal that may have leading and rear fronts. Since the world time  $\mathcal{T}$  (4.9) is a harmonic function it can be discontinuous along characteristics. The time of emission of a photon, in principle, is not defined. A photon does not constitute a signal – the operators of the electric field and of the number of photons do not commute with each other and cannot have common eigenfunctions.

What if it occurred possible to trace an observed radiation field back to the current in the interior of the localized object (so that  $J_\mu = j_\mu$ ) and, e.g. by a precise analysis of radiation, to learn that  $Q_{\mu\nu} \neq 0$  there? Then Eq.(4.1) must be replaced by the second equation of (6.10). Proceeding as previously, we get

$$\nabla_\mu e_\nu^{(0)} - \nabla_\nu e_\mu^{(0)} + e_\nu^{(0)} \partial_\mu \ln \mathcal{R} - e_\mu^{(0)} \partial_\nu \ln \mathcal{R} = -(1/\mathcal{R}) Q_{\mu\nu}. \quad (6.11)$$

Contracting this equation with spacelike  $e_{(i)}^\nu e_{(j)}^\mu$  ( $i, j = 1, 2, 3$ ) and recalling Eqs.(2.19) we find that

$$\omega_{0ij} - \omega_{0ji} = -(1/\mathcal{R}) Q_{ij}. \quad (6.12)$$

If  $Q_{ij} \neq 0$  starting from some time moment  $t_0$ , then at  $t > t_0$  the congruence  $e_{(0)}^\mu$  of lines of the vector current cannot be a normal congruence. The family of space-like surfaces  $t = \text{const}$ , orthogonal to the vector current, vanishes. This means that Eq.(4.3) cannot be obtained and Dirac field cannot form a stable object<sup>11</sup>. The electromagnetic fields produced by such an object are not just longitudinal (Coulomb) fields and the object must start to radiate solely because the electromagnetic field around it is not simultaneous with its source. Since the Dirac equation is of the hyperbolic type, the changes of the Dirac field must propagate also, having a light cone as a leading wave front.

Contracting Eq.(6.11) with  $e_{(0)}^\nu e_{(i)}^\mu$  we obtain another equation,

$$\omega_{i00} = -(\partial \ln \mathcal{R} / \partial s_i) - (1/\mathcal{R}) Q_{i0}, \quad (6.13)$$

that accounts for the effect of the electric field, which adds a boost in the direction of the congruence  $e_{(i)}^\mu$ . Interaction with the electric field alone (which can be the case only when this field is longitudinal) does not destroy the hypersurfaces of constant time of a localized object, which allows it to stay intact. The most important effect of acceleration in an electric field is altering the shape of a charged object which leads to an increase of its *internal energy* and of the local charge density.

Referring to the above qualitative analysis and analysis of solutions that admit the lightlike currents (in Sec. VII), we may go further and discuss a qualitative picture of some transient processes. If the Dirac waveform is not stable (as is in the case of  $\mu^+$ ) then

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<sup>11</sup> This is yet another way to view two seemingly different phenomena, the Meissner effect and precession of the spin in magnetic field. In order to be in a stable quantum state, the superconductor expels magnetic field from its interior or confines it into vortices, thus defining a common time across its whole volume. In the same way, electron with the magnetic moment, being placed in magnetic field, moves in precession with the Larmor frequency. Therefore, in rotating frame the magnetic field vanishes and the electron still can have the same world time across its volume (staying in a certain quantum state).

the development of instability (and the lifetime) must still be stretched, due to the nonlinear effect of the time slowdown. When the limit of stability (bifurcation) at  $r \sim \lambda_\mu = 1.86 \cdot 10^{-13} \text{cm}$  is reached, then the previous dynamic regime suddenly breaks up and the field begins to evolve towards a new configuration of a smaller mass  $m_e$  and a larger  $\lambda_e = 3.86 \cdot 10^{-11} \text{cm}$ . The Dirac field of the  $\mu^+$ , which originally was localized near the caustic, must be radiated. Since the Dirac equation is hyperbolic, the sharp front of the Dirac field, as any *precursor*, must propagate along a characteristic (at the speed of light), in the outward direction. This can only be the right component of the Dirac spinor (which inherits its lightlike current from the caustic), the  $\bar{\nu}_\mu$ . The final state of a  $e^+$  emerges not earlier than a new caustic at  $r = \lambda_e$  is formed. This requires yet another transient process of the violent collapse of the Dirac matter onto a new caustic and radiation of a similar precursor, but with the opposite chirality,  $\nu_e$ . The leptonic number is preserved dynamically in both processes. Remarkably, it is exactly the existence of a well-defined (by the spacelike *axial* vector) outward direction that eliminates the illusion of the reflection symmetry of a plane wave and thus predetermines a unique polarization of the spinor precursors. As it was noted by Wigner [20], it is only a theoretical idea of mirror symmetry (expressed in terms of *polar* vectors) that hints of the possible existence of the second polarization for lightlike spinor waveforms.

## VII. SINGULAR TIMELIKE CURRENTS, MAJORANA CONDITION AND NEUTRINOS-PRECURSORS.

So far, considering the spacetime metric as a descendant of the material Dirac field, we always assumed that  $\mathcal{R}^2 > 0$ . In this case, the vector current  $j^\mu$  is timelike and the axial current  $\mathcal{J}^\mu$  is spacelike, so that (after a special choice of the local tetrad) both currents can have only one nonzero component. The case of  $\mathcal{R}^2(x) = 0$  is special because wherever  $j^2(x) \equiv -\mathcal{J}^2(x) = 0$  each of these vectors must have either none or at least two nonzero components. Such a qualitative change is impossible without discontinuity at least of the derivatives of the Dirac field. The orthogonal local tetrad must degenerate into a smaller set of the lightlike vectors. Since along the isotropic lines we have  $ds = 0$ , the Fermi-Walker transport along  $j^\mu$  or  $\mathcal{J}^\mu$  becomes impossible. If it happens as the result of a physical process then this singular behavior must be inherited by the matter-induced metric. By virtue of Eqs.(4.10) and (4.11) we will also have  $g^{00} = 0$ , so that d'Alembert equation (4.9) cannot be the equation for the time variable. The Dirac field would never form localized objects if it had  $\mathcal{R}^2 = 0$  everywhere. At best, the limit of  $\mathcal{R}^2 = 0$  can be reached on singular surfaces (actually, the wave fronts = the shock waves). On these surfaces, the lightlike  $j^\mu$  cannot even be interpreted as a conserved current.

There are two reasons to look at the singular case of  $\mathcal{R}^2 = 0$  in details.

(i) This condition holds on the wave front of the Dirac field, which bears information about arrival of a signal. The structure of the field behind the leading front carries even more information about the nature of the transient process that initiated the propagating discontinuity. The question about physical effects it can produce is the most important one.

(ii) The limit  $\mathcal{R}^2 = 0$  is met under the so-called Majorana additional condition; hence, it is connected with the problem of existence of the massive neutral Dirac particle (the Majorana neutrino) and of the neutrinoless double  $\beta$ -decay<sup>12</sup>. In the context of the Dirac waveforms, the hope that such a process can exist is connected with the polarization properties of neutrinos considered as the precursors.

To address these two issues, let us notice that in terms of the components of the Dirac spinor,  $\psi(x) = (u_L, d_L, u_R, d_R)$ , the condition  $\mathcal{R}^2 = 0$  reads as

$$\mathcal{R}^2 = 4(u_R^* u_L + d_R^* d_L)(u_L^* u_R + d_L^* d_R) = 0. \quad (7.1)$$

Being presented in this form,  $\mathcal{R}^2$  obviously is the squared modulus of the complex number,  $2(u_R^* u_L + d_R^* d_L)$ ; therefore, it is equivalent to

$$u_R^* u_L + d_R^* d_L = 0. \quad (7.2)$$

By virtue of the identities (2.8), the last condition holds only in the singular domains where the vector and axial currents of the Dirac field are lightlike,  $j^2 = \mathcal{J}^2 = 0$ , and the tensor of polarization,  $\mathcal{M}^{ab}$ , has the structure of the transverse plane wave,  $\vec{L}^2 - \vec{K}^2 = \vec{L} \cdot \vec{K} = 0$ .

The detailed calculations of the shape of precursors (similar to those for electromagnetic precursors [21]) requires simultaneous account for the effects of propagation and for transient process in the source, which is a difficult problem. However, it is possible to obtain some useful information by considering the limit of  $\mathcal{R}^2 \rightarrow 0$  (i.e., by approaching the singular surface from behind the leading front). For this purpose, it is expedient to rewrite the complex equations (7.2) in terms of absolute values and phases of the spinor components,

$$u_{(R)} = |u_{(R)}| \exp[i\varphi_{(R)}], \quad d_{(R)} = |d_{(R)}| \exp[i\chi_{(R)}].$$

The complex equation (7.2) is equivalent to the system

$$\begin{aligned} \varphi_R - \chi_R &= \varphi_L - \chi_L \mp \pi, \\ |u_R||u_L| &= |d_R||d_L|. \end{aligned} \quad (7.3)$$

Now, the limit of  $\mathcal{R}^2 \rightarrow 0$  can be approached gradually, by employing the phase relations of (7.3) as the first step.

<sup>12</sup> I am indebted to Prof. Vladimir Zelevinsky for pointing this out to me.

In this way, we will be able to determine the quantities that characterize the field of the shock wave as well as its possible effect on stable matter.

By the definition of scalar densities, we have

$$\mathcal{P} = i(u_R^* u_L + d_R^* d_L - u_L^* u_R - d_L^* d_R) \quad (7.4)$$

$$\begin{aligned} &= 2(|u_R||u_L| \sin(\varphi_L - \varphi_R) + |d_R||d_L| \sin(\chi_L - \chi_R)) \\ \mathcal{S} &= (u_R^* u_L + d_R^* d_L + u_L^* u_R + d_L^* d_R) \quad (7.5) \\ &= 2(|u_R||u_L| \cos(\varphi_L - \varphi_R) + |d_R||d_L| \cos(\chi_L - \chi_R)). \end{aligned}$$

The last two equations allow one to write down the invariant  $\mathcal{R}^2$  as

$$\begin{aligned} \mathcal{R}^2 &= \mathcal{P}^2 + \mathcal{S}^2 = 4[|u_R|^2 |u_L|^2 + |d_R|^2 |d_L|^2 \\ &+ 2|u_R||u_L||d_R||d_L| \cos(\varphi_L - \varphi_R - \chi_L + \chi_R)], \quad (7.6) \end{aligned}$$

so that, using the first of Eqs.(7.3), we arrive at

$$\begin{aligned} \mathcal{P} &= 2(|u_R||u_L| - |d_R||d_L|) \sin(\varphi_L - \varphi_R), \\ \mathcal{R} &= 2(|u_R||u_L| - |d_R||d_L|). \quad (7.7) \end{aligned}$$

Now it is evident that in the limit determined by both equations (7.3) we have  $\mathcal{P} = 0$  and  $\mathcal{S} = 0$ , but the ratio

$$\sin \Upsilon = \mathcal{P}/\mathcal{R} = \pm \sin(\varphi_L - \varphi_R) \quad (7.8)$$

is a finite number. The propagating pulse emitted in the course of a transient process has a small but finite width and its distinctive feature is a sudden phase shift (7.8) between left and right components of the Dirac field<sup>13</sup>. By a simple geometric argument one can show that, precisely on the leading front, the composition of the two vector currents,  $j^\mu$  and  $\mathcal{J}^\mu$ , becomes *either* right *or* left lightlike current. None of these currents is conserved and, according to Eqs. (3.4), they have the pseudoscalar density as the source (sink). The two vectors that are used to form the local orthogonal tetrad, one timelike and one spacelike, degenerate into one lightlike current with two equal components. The vector current that initially had only one positive time component acquires the second spatial component by merging with the axial current. Since the two nonzero components of the left and right currents coincide (modulo the sign of the space component), Eqs. (3.4) become the equations for kinematic waves and can be integrated along their characteristics. In fact, we are dealing with the phenomenon of the restoration (for a short instance) of the symmetry which is broken by the presence of the localized objects (the inward and outward directions were distinct). Exactly on the leading front of the transient process, this information must be lost; indeed, the shock wave can be created only in the course

of the interaction, which is responsible for a sudden reshaping of a localized object.

The remaining subtle issue is whether the pulse with the lightlike wave front should be associated with neutrino as neutral or charged, massive or massless particle. This is the subject of the longstanding controversy around the nature of Weyl, Dirac, and Majorana neutrino [23]. The congruences with lightlike tangent vectors are the characteristics of the hyperbolic system of the Dirac equations. The net of characteristics densely covers the entire space; of a special physical significance are only those of them, along which the Dirac field is discontinuous. These characteristics serve as the fronts of the propagating signals. The actual questions are about the length of these signals and the structure of the Dirac field behind the leading front and not about the mass parameter that could have been assigned to the precursor treated as a particle. The answers can only be obtained from the nature of the transient process that initiates the propagating discontinuity. It is logical to base the crudest classification –  $\nu_e$ ,  $\nu_\mu$ ,  $\nu_\tau$  – on the size of the object which is created or decays. The transient process of the smallest object must be the shortest one. The short pulses of the left and right currents always have the  $V - A$  structure, which probably explains the incredible accuracy of the  $V - A$  scheme in the description of the *basic* weak interactions (the Weyl neutrino of the Standard Model and of the old theory of the four-fermion interaction).

A refined characterization should include the explicit shape of the pulses which then may belong to continuous spectrum of the almost lightlike waveforms. Behind the leading front, these pulses must have both left and right components, thus being qualitatively close to the Dirac neutrino. The physical effect of the leading front (a sudden phase shift between the left and right spinor components) can be explosive; according to Eqs. (4.20), the stability of the localized objects is sensitive to fine tuning of these phases. When such a front crosses a localized object it can cause a reaction. While propagation of the neutrino signals in matter is a classical process similar to propagation of the electromagnetic precursors [21], the quantum aspects of reactions they may induce are governed by the detectors [24].

Accepting the nature of neutrino as coherent precursors of a “moderately hard” processes, one must expect a further hardening in the course of propagation in matter. For example, the spectrum of  $\nu_e$  should gradually drift into domain of higher frequencies. In general, the spectrum of precursor is broad but (for the electromagnetic precursors) the local frequency at the distance  $d$  is proportional to  $\sqrt{d}$  [21]. At certain distances from the emitter it will interact with matter similarly to precursors created in harder processes ( $\nu_\mu$  or  $\nu_\tau$ ). Outside these “resonant” periods of their life these pulses may be associated with the sterile neutrinos or weakly interacting massive particles.

The issue of the Majorana neutrino is the most controversial one. In order to discuss it in the framework of

<sup>13</sup> This phase shift qualitatively resembles the infinitely thin front of the transverse electromagnetic radiation arising when an instantaneous change of the parameters of a system of charges occurs. Ahead and behind the front of radiation the field is the longitudinal Coulomb field of initial and final configurations of charges, respectively [22].

waveforms, let us notice that Eq. (7.2) has a special solution, which is a linear relation between the components of  $\psi$  and its complex conjugate  $\psi^*$ ,

$$u_R = d_L^*, \quad d_R = -u_L^*. \quad (7.9)$$

In the matrix form, this condition reads as

$$\psi^c \equiv \mathbf{C}\psi^* = \psi, \quad \mathbf{C} = \rho_2 \sigma_2 = \begin{pmatrix} 0 & -i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}, \quad (7.10)$$

where  $\mathbf{C}$  is the well-known matrix of the charge conjugation in the spinor representation. The formulae (7.9-10) are known as the Majorana *additional condition*, under which the massive Dirac particle is supposed to be neutral [23].

Being *ad hoc* imposed on the Dirac field, without reference to the origin of the condition (7.2) as the light-front limit of a transient process, the Majorana condition enforces the light-front behavior in the entire spacetime. It does not allow one to smoothly approach the leading front (the right hand sides of Eqs.(7.4-7) immediately become zero). Being imposed *after* the Jordan-Wigner quantization, the Majorana condition leads to the conclusion that the vector current of the Dirac field, which is regarded as the electric current, is the identical zero. One cannot reach this zero smoothly, preserving the identities (2.8). It never has been noticed that under the Majorana condition (despite the finite mass in the equation of motion) the vector current becomes lightlike, left or right, even before quantization; it is not conserved anymore and it cannot represent the conserved electric current neither before nor after quantization. In the context of the charges as localized waveforms the Dirac field is not quantized in terms of plane waves and the sign of electric charge is associated with the direction of the axial current and the sign of the pseudoscalar density. Under condition (7.9), the classical axial current becomes zero, while the vector current remains finite and singles out the future light cone.

The view of the Dirac particles as the localized waveforms creates a new framework for the investigation of the controversial issue of the neutrinoless double  $\beta$ -decay [23]. The presence of both left and right components in the field of the precursors seemingly does not prohibit this process. The question is, however, if an interplay of the right and left components can lock the entire transient process inside the nucleus (and suppress the emission of neutrinos during simultaneous emission of two electrons). It has no simple answer and requires a detailed investigation of the spacetime picture of the whole process.

## VIII. CONCLUSION.

The nonlinear Dirac equation, with its capricious interplay of the many polarization degrees of freedom, poses a tough mathematical challenge for theory. Its explicit

solutions may well yield various "magic numbers" that are currently known only from experiment. Even before regular mathematical methods are developed, one may rely on various qualitative consequences of the finite size of the Dirac waveforms to re-analyze existing data.

1. The conjectured connection between the mechanism of self-localization and the sign of the electric charge of the Dirac wave form also assumes that positively charged particles, which are not perfectly stable, must have a somewhat longer lifetime than their negatively charged anti-particles. By the same argument, positively charged particles must have somewhat smaller magnetic moment (or gyromagnetic ratio). The ratios  $\chi = (\tau_+ - \tau_-)/\tau_{av}$  and  $\xi = (g_+ - g_-)/g_{av}$  were measured for the most long-lived species as a test of CPT-invariance. According to the Particle Data Group [17], the difference in lifetime is indeed always positive,  $\chi(K^\pm) = (0.11 \pm 0.09) \cdot 10^{-2}$ ,  $\chi(\pi^\pm) = (5.5 \pm 7.1) \cdot 10^{-4}$ ,  $\chi(\mu^\pm) = (2 \pm 11) \cdot 10^{-5}$ , being the largest for the heaviest specie. Similarly,  $\xi(e^\pm) = (-0.5 \pm 2.1) \cdot 10^{-12}$ ,  $\xi(\mu^\pm) = (-0.11 \pm 0.12) \cdot 10^{-8}$ , and  $(\mu_p + \mu_{\bar{p}})/\mu_p = (-2.6 \pm 2.9) \cdot 10^{-3}$ . Though this data have a low accuracy, the trend is stable.

2. One of the predicted manifestations of charge asymmetry is the existence of the particle's external size. The internal radius is universally limited from the below by the Compton length  $\lambda = \hbar/mc$ . Due to the time slowdown in domains of large Dirac density, the positively charged species must have smaller external size than their negatively charged partners. Possibly,  $e^+$  has reasonably well defined external boundary, which then may explain its relatively long lifetime in the environment of normal matter. There may well exist observed differences in the dynamics of electrons and positrons (or  $p$  and  $\bar{p}$ , e.g., in storage rings) that are currently attributed to technical issues. The aforementioned difference in magnetic moments of  $e^+$ ,  $p$  and  $e^-$ ,  $\bar{p}$  may result in different intensity of their synchrotron radiation and a longer time of acquiring stable mode for  $e^-$  and  $\bar{p}$  when radiation losses are equally compensated.

3. There are certain coincidences of numbers that may prompt another look at well known phenomena. We know that,  $\lambda_e = 386\text{fm}$ ,  $\lambda_\mu = 1.86\text{fm}$ . The radius of the proton, as estimated via its electromagnetic form-factors, is  $r_p = 1\text{fm}$  and it grows for nuclei as  $A^{1/3}$ . This may well tell us something about the high rate of  $\mu^-$  capture by light nuclei versus the low rate of the inverse  $\beta$ -decay by even heavy nuclei. The correlation between the capture rate and size of a nucleus may be a useful test.

4. The failure to keep anti-hydrogen molecule in the cold atom trap for an indefinitely long time may establish the limits of stability of the antiproton in antimatter surroundings. One may think of the capture of  $e^+$  by the imperfectly localized  $\bar{p}$  with a subsequent decay into pions as a possible mechanisms of instability.

5. In Sec.VIA we have mentioned that the dynamics of the  $K^0 \leftrightarrow \bar{K}^0$  oscillations may result in the difference between the asymmetry parameters,  $\Delta A_L = A_L(e) -$

$A_L(\mu) \neq 0$ . Both  $A_L(e)$  and  $A_L(\mu)$  were measured in Ref. [28], but with a relatively low accuracy. The current estimate is  $\Delta A_L = (+0.030 \pm 0.026)\%$  (slightly above  $1\sigma$ ). For  $A_L(e)$  much higher accuracy ( $\pm 0.006$ ) was obtained in the recent KTeV experiment [29]. A new measurement of the  $A_L(\mu)$  with a comparable precision is desirable.

6. The Lorentz contraction of the accelerated waveforms is a dynamic effect, which leads to the accumulation of energy in an extremely small volume and its release in the course of a collision. The 5-8 GeV electrons and positrons are compressed to a size about  $10^{-1}\text{fm}$  having a density higher than a proton. Upon colliding, they stop and create a sharp peak of invariant density, which is very far from a stable configuration and rapidly decays. The  $B^\pm$  lifetime is reasonably long,  $1.7 \cdot 10^{-12}\text{s}$ , and its size is about  $10^{-1}\text{fm}$ . The time slowdown at  $\mathcal{R} \gg 1$  may result in two effects: (i) an abnormally long lifetime of the resonance and (ii) an exotic trend to further decay into compact heavy objects rather than to decay into lighter objects according to the usual spectator model. The mode  $B^+ \rightarrow K^+ X(3872) \rightarrow J/\psi \pi^+ \pi^-$  seems to be a candidate for this kind of the process because the width of  $X(3872)$  is very small.

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### Appendix A: Parallel transport of Dirac field.

In order to derive a measure for comparison of the fields  $\psi(x_1)$  and  $\psi(x_2)$  at two close points let us require, following Fock [3], that the components  $j^a = \psi^+ \alpha^a \psi$  are the invariants of the vector  $j^\mu(x)$  and the congruences  $e_{(a)}^\mu(x)$  at the point where vector is defined,  $j^a = e_{(a)}^\mu j^\mu$ . For now, we assume that a set of four orthogonal congruences is fixed in advance and that Dirac matrices  $\alpha^a$  are either invariants or covariantly constant objects. When the invariant  $j_a$  is parallel-transported by  $ds_b$  along an arc of congruence (b), then, solely because the local pyramid is being rotated, it must change by  $\delta j_a = \omega_{acb} j^c ds^b = \omega_{acb} \psi^+ \alpha^c \psi ds^b$ . The invariants  $\omega_{abc}$  are defined by Eq.(2.15). Let matrix  $\Gamma_a$  (the connection) define the change of the Dirac field components in the course of the same infinitesimal displacement,  $\delta \psi = \Gamma_a \psi ds^a$ ,  $\delta \psi^+ = \psi^+ \Gamma_a^+ ds^a$ . Let differential of

the product  $\psi^+ \alpha^a \psi$  obey Leibnitz rule. This gives yet another expression for  $\delta j_a$ ,

$$\delta j_a = \psi^+ (\Gamma_b^+ \alpha_a + \alpha_a \Gamma_b) \psi ds^b. \quad (\text{A.1})$$

The two forms of  $\delta j_a$  must be identical. Hence, the equation that defines  $\Gamma_a$  is

$$\Gamma_b^+ \alpha_a + \alpha_a \Gamma_b = \omega_{acb} \alpha^c, \quad (\text{A.2})$$

and it has the most general solution,

$$\Gamma_b(x) = ieA_b(x) + ig\rho_3 \aleph_b(x) - \frac{1}{2} \omega_{0kb}(x) \rho_3 \sigma_k - \frac{i}{4} \epsilon_{0kim} \omega_{imb}(x) \sigma_k, \quad (\text{A.3})$$

where the last two terms can be compacted as,  $\Omega_b = (1/4) \omega_{cdb} \rho_1 \alpha^c \rho_1 \alpha^d = (1/4) \omega_{cdb} \gamma^c \gamma^d$ . These two terms correspond to an infinitesimal boost of (2.4) along the spatial  $k$ -axis with parameter  $\omega_{0kb} ds^b$  and an infinitesimal rotation of (2.3) in the  $(im)$ -plane with parameter  $\omega_{imb} ds^b$ , respectively. This analogy, however, is limited. While Eqs.(A.1)-(A.3) do imply some measure for the length of an arc (and of an angle as the ratio of the two lengths), Eq.(2.2) does not. The first two terms are due to an intrinsic indeterminacy that arises when one has to compare Dirac fields at two different points relying only on the properties of the vector forms  $e_{(a)}^\mu(\psi)$ . The first term is readily associated with the electromagnetic potential. The second one would not appear at all if, following Fock [3], we required that  $\delta(\psi^+ \rho_1 \psi) = 0$  and  $\delta(\psi^+ \rho_2 \psi) = 0$ . This decision was motivated by that kind of invariance of the Dirac equation in Minkowski space (local Lorentz invariance), which is not inherited by the Dirac field in Riemannian geometry. The position of  $\aleph_b$  in connection (A.3) may lead to the impression that it can well be a “next field”, which interacts with the axial current  $\mathcal{J}_b$  of the Dirac field and is governed by an independent equation of motion. At least for the stable configurations of the Dirac field, this is not true.

The connection (A.3) commutes with the matrix  $\rho_3$  so that Eq.(A.2) remains the same when  $\alpha_a \rightarrow \rho_3 \alpha_a$ . It neither commutes nor anti-commutes with  $\rho_1$  and  $\rho_2$ , viz.

$$\begin{aligned} \Gamma_b^+ \rho_1 + \rho_1 \Gamma_b &= 2g\rho_2 \aleph_b, \\ \Gamma_b^+ \rho_2 + \rho_2 \Gamma_b &= -2g\rho_1 \aleph_b. \end{aligned} \quad (\text{A.4})$$

From now on, we postulate that invariant derivative of the Dirac field is  $D_a \psi = (\partial_a - \Gamma_a) \psi$  where  $\partial_a = e_a^\mu \partial_\mu$  is the derivative in the direction of a curve of congruence (a). Assuming the Leibnitz rule for  $D_a$  and considering all Dirac matrices as constants we readily reproduce the reference point of Eqs.(A.1) and (A.2) as

$$D_b j_a = \partial_b j_a - \omega_{acb} j_c \equiv \nabla_b j_a. \quad (\text{A.5})$$

The result (A.5) for  $D_b j_a$  is a warrant that after projecting the r.h.s. into coordinate space we must recover

the covariant derivative  $\nabla_\mu j_\nu$ <sup>14</sup>. Indeed,  $e_\mu^a e_\nu^b \nabla_b j_a = \partial_\mu j_\nu - \Gamma_{\nu\mu}^\sigma j_\sigma = \nabla_\mu j_\nu$ , and we shall consider this as a proof that  $j_a$  is an invariant of the vector  $j_\mu$  and congruence  $e_{(a)}^\mu$ .

In exactly the same way we may verify that the invariants  $D_b \mathcal{J}_a$  of the axial current are of the form  $D_b \mathcal{J}_a = \nabla_b \mathcal{J}_a$  and conclude that  $e_\mu^a e_\nu^b D_b \mathcal{J}_a = \nabla_\mu \mathcal{J}_\nu$ . *Vice versa*, the quantities  $D_b \mathcal{J}_a = e_\mu^a e_\nu^b \nabla_\mu \mathcal{J}_\nu$  are invariants of a tensor and a system of congruences. It is straightforward to verify (computing all derivatives as functions of  $\psi$ ) that equations like (A.5) hold not only for  $j^\mu$  and  $\mathcal{J}^\mu$  but for  $e_{(0)}^\mu[\psi] = j^\mu/\mathcal{R}$  and  $e_{(3)}^\mu[\psi] = \mathcal{J}^\mu/\mathcal{R}$ . Recalling discussion of Sec.II, we may view this as complementary to (2.14), i.e., proof that the unit vectors  $e_{(0)}^\mu$  and  $e_{(3)}^\mu$  are vectors of Riemannian geometry. For the same reason, the projectors  $[\delta_\nu^\mu - j^\mu j_\nu/\mathcal{R}^2]$  and  $[\delta_\nu^\mu + \mathcal{J}^\mu \mathcal{J}_\nu/\mathcal{R}^2]$  are the tensors.

Using the same technique of differentiating and by virtue of Eqs.(A.4) we obtain

$$D_a \mathcal{S} = \partial_a \mathcal{S} - 2g\mathcal{P}\aleph_a, \quad D_a \mathcal{P} = \partial_a \mathcal{P} + 2g\mathcal{S}\aleph_a. \quad (\text{A.6})$$

As one can see, that there is no immediate correspondence between the algebraic and differential properties of the scalars. However, the quantities from the first line of (2.8) (like  $\mathcal{R}$ ) are differentiated as true scalars. The same behavior is observed for the components of the skew-symmetric tensor  $\mathcal{M}_{ab}$ . Instead of the anticipated  $D_c \mathcal{M}_{ab} = \nabla_c \mathcal{M}_{ab} \equiv e_c^\lambda e_a^\mu e_b^\nu \nabla_\lambda \mathcal{M}_{\mu\nu}$  we encounter one more disagreement with the differential criterion (A.5),

$$\begin{aligned} D_c \mathcal{M}_{ab} &= \nabla_c \mathcal{M}_{ab} - 2g\aleph_c^* \mathcal{M}_{ab}, \\ D_c^* \mathcal{M}_{ab} &= \nabla_c^* \mathcal{M}_{ab} + 2g\aleph_c \mathcal{M}_{ab}. \end{aligned} \quad (\text{A.7})$$

We leave open the question of if and when  $e_{(1)}^\mu$  and  $e_{(2)}^\mu$  of Eqs.(2.7) are the vectors with the same degree of confidence as  $e_{(0)}^\mu$  and  $e_{(3)}^\mu$ . In most cases,  $e_{(1)}^\mu$  and  $e_{(2)}^\mu$  correspond to angular coordinates, which are physically uncertain without external fields (other objects nearby). The expected one-to-one match with geometry is spoiled by the extra (with respect to generator of Lorentz transformations in the connection  $\Gamma_a$ ) matrices  $\rho_1$  and  $\rho_2$  responsible for the “mixing” between right and left components. From this perspective, the Sakharov’s idea [2], regarding the topological nature of elementary charges, seems be closer to reality that it initially appeared.

Recalling Eqs.(2.10) and using (A.6) we can compare the covariant derivative  $D_a \mathcal{P}$  computed in two ways, as  $D_a \mathcal{P} = \partial_a(\mathcal{R} \cdot \sin \Upsilon) + 2g\mathcal{R} \cos \Upsilon \aleph_a$  or, alternatively, as  $D_a \mathcal{P} = \partial_a \mathcal{R} \cdot \sin \Upsilon + \mathcal{R} \cos \Upsilon D_a \Upsilon$ . The result reads as

$$D_a \Upsilon[\psi] = \partial_a \Upsilon + 2g\aleph_a, \quad (\text{A.8})$$

which is invariant under the simultaneous transformations,  $\Upsilon \rightarrow \Upsilon + Y(x)$  and  $2g\aleph_a \rightarrow 2g\aleph_a - \partial_a Y(x)$ . A supposed freedom of such (chiral) transformations is not permissible, since these transformations change the observables in the r.h.s. of Eqs.(3.13) and (B.4) without altering the l.h.s.

The commutator  $[D_a, D_b]$  still contains derivatives. Indeed,

$$\begin{aligned} [D_a, D_b]\psi &= (\partial_a \partial_b - \partial_b \partial_a)\psi \\ &\quad - [\partial_a \Gamma_b - \partial_b \Gamma_a - \Gamma_a \Gamma_b + \Gamma_b \Gamma_a]\psi. \end{aligned} \quad (\text{A.9})$$

However, for practical purposes it is important that  $[D_a, D_b]\psi$  can be split into two parts, with and without derivatives. Since  $\psi$  is a coordinate scalar and, in general, derivatives along arcs do not commute, we have  $(\partial_a \partial_b - \partial_b \partial_a)\psi = (\omega_{cab} - \omega_{cba})\partial_c \psi$ . Now we can reassemble  $[D_a, D_b]\psi$  as follows,

$$\begin{aligned} [\vec{D}_a, \vec{D}_b]\psi &= (\omega_{cab} - \omega_{cba})\vec{D}_c \psi - \mathbb{D}_{ab}, \\ \mathbb{D}_{ab} &= [\partial_a \Gamma_b - \partial_b \Gamma_a - \Gamma_a \Gamma_b + \Gamma_b \Gamma_a - C_{cab} \Gamma_c]\psi, \end{aligned} \quad (\text{A.10})$$

where the term  $C_{cab} \Gamma_c \equiv (\omega_{cab} - \omega_{cba})\Gamma_c$  is added and subtracted to replace  $\partial_c \psi$  by the covariant derivative  $D_c \psi$ . The matrix operator  $\mathbb{D}_{ab} = -e_a^\mu e_b^\nu [D_\mu, D_\nu]$  in the r.h.s. does not contain derivatives and can be explicitly found,

$$\begin{aligned} \mathbb{D}_{ab} &= -\frac{1}{4} R_{abcd} \rho_1 \alpha^c \rho_1 \alpha^d + ie F_{ab} + ig \rho_3 U_{ab}, \quad (\text{A.11}) \\ F_{ab} &= \partial_a A_b - \partial_b A_a - (\omega_{cab} - \omega_{cba}) A_c, \\ U_{ab} &= \partial_a \aleph_b - \partial_b \aleph_a - (\omega_{cab} - \omega_{cba}) \aleph_c, \end{aligned}$$

where invariants of the Riemann curvature tensor are defined by Eq.(2.23) and  $F_{ab}$  and  $U_{ab}$  are invariants of the electromagnetic tensor  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$  and the field tensor  $U_{\mu\nu} = \nabla_\mu \aleph_\nu - \nabla_\nu \aleph_\mu$ , respectively.

Using the cyclic symmetry of the Riemannian curvature tensor it is straightforward to show [3] that

$$\alpha^a \mathbb{D}_{ab} = \frac{1}{2} \alpha^a R_{ab} + ie \alpha^a F_{ab} + ig \rho_3 \alpha^a U_{ab}. \quad (\text{A.12})$$

When field  $\aleph_a$  in the connection  $\Gamma_a$  is a gradient, the tensor  $U_{\mu\nu}$  vanishes identically, which is assumed throughout this paper except for Eq.(B.10).

<sup>14</sup> Schouten ([8], Ch.II, §9; Ch.III, §9) considers equations like (A.5) as a condition that fixes the components of a vector with respect to nonholonomic coordinate system.

## Appendix B: Stress tensor and pseudoscalar field.

### 1. Internal flux of mass and stress in the Dirac field.

In this section we study the stress tensor  $P_b^a = i\psi^+ \rho_3 \alpha^a D_b \psi$ , mostly following the same logic as for the energy momentum tensor  $T_b^a = i\psi^+ \alpha^a D_b \psi$  in Sec.IIIB, starting from its covariant derivative. We find that

$$D_c[\psi^+ \rho_3 \alpha^a \vec{D}_b \psi] = \partial_c[\psi^+ \alpha^a \vec{D}_b \psi] - \omega_{adc} \psi^+ \rho_3 \alpha^d \vec{D}_b \psi. \quad (\text{B.1})$$

Once again, the last term of Eq.(3.5) is missing, and thus we have no confidence that the covariant derivative is a tensor. This time, let us begin by contracting indices  $a$  and  $b$  in Eq.(B.1),

$$D_c[\psi^+ \rho_3 \alpha^a \vec{D}_a \psi] = \partial_c[\psi^+ \rho_3 \alpha^a \vec{D}_a \psi] - \omega_{abc} \psi^+ \rho_3 \alpha^b \vec{D}_a \psi. \quad (\text{B.2})$$

By virtue of the Dirac equations, the first term in the r.h.s. of (B.2) becomes  $\partial_c[m\psi^+ \rho_2 \psi]$ . Alternatively, we can immediately use the equations of motion in the l.h.s. and only then differentiate (matrices  $\rho_3$  and  $\alpha^a$  commute),

$$D_c[\psi^+ \rho_3 \alpha^a \vec{D}_a \psi] = mD_c[\psi^+ \rho_2 \psi] = m\partial_c[\psi^+ \rho_2 \psi] + m \cdot 2g\mathcal{S}\mathcal{N}_c. \quad (\text{B.3})$$

Comparing the last two equations we finally get the equation,

$$\omega_{acb} \cdot P_{ca} = -2igm\mathcal{S}\mathcal{N}_b, \quad (\text{B.4})$$

which is complementary to Eq.(3.13). The imaginary part in the l.h.s. is due to  $(1/2)[P_{ca} - P_{ca}^+] = (i/2)D_c\mathcal{J}_a$ . Since the axial current is a vector, we can rewrite the last equation as

$$(1/2)\omega_{acb}\nabla_c\mathcal{J}_a = -2gm\mathcal{S}\mathcal{N}_b, \quad (\text{B.5})$$

which is complementary (dual) to Eq.(4.19). The skew-symmetric Hermitian part,  $(P_{ca} + P_{ca}^+) - (P_{ac} + P_{ac}^+)$ , must vanish since the r.h.s. of Eq.(B.4) is an imaginary quantity. This yields the equation, which duplicates Eq. (4.1),

$$i[\psi^+ \rho_3 \alpha_a \vec{D}_c \psi - \psi^+ \overleftarrow{D}_c^+ \alpha_a \rho_3 \psi - \psi^+ \rho_3 \alpha_c \vec{D}_a \psi + \psi^+ \overleftarrow{D}_a^+ \alpha_c \rho_3 \psi] = \epsilon_{acut} D_u j_t = 0. \quad (\text{B.6})$$

and thus indicates that we still are dealing with a stable waveform.

Contracting (in Eq.(B.2)) indices  $a$  and  $c$  we arrive at the expression, which is similar to Eq.(3.8),

$$D_a[\psi^+ \rho_3 \alpha^a \vec{D}_b \psi] = \partial_a[\psi^+ \rho_3 \alpha^a \vec{D}_b \psi] + \omega_{acc} \psi^+ \rho_3 \alpha^a \vec{D}_b \psi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{-g} e_{(a)}^\nu (\psi^+ \rho_3 \alpha^a \vec{D}_b \psi) \right]. \quad (\text{B.7})$$

Let us first rewrite the l.h.s. of Eq. (B.7) as

$$D_a[\psi^+ \rho_3 \alpha^a \vec{D}_b \psi] = \psi^+ \rho_3 \alpha^a [\vec{D}_a \vec{D}_b - \vec{D}_b \vec{D}_a] \psi + \psi^+ \overleftarrow{D}_a^+ \rho_3 \alpha^a \vec{D}_b \psi + D_b(\psi^+ \rho_3 \alpha^a \vec{D}_a \psi) - \psi^+ \overleftarrow{D}_b^+ \rho_3 \alpha^a \vec{D}_a \psi. \quad (\text{B.8})$$

Because of an obvious change of signs (caused by an extra  $\rho_3$ ), the last three terms of (B.8) do not cancel. Instead of (3.9) we have

$$D_a P_b^a = i\psi^+ \rho_3 \alpha^a [\vec{D}_a \vec{D}_b - \vec{D}_b \vec{D}_a] \psi + imD_b \mathcal{P} + im[\psi^+ \rho_2 \vec{D}_b \psi - \psi^+ \overleftarrow{D}_b^+ \rho_2 \psi]. \quad (\text{B.9})$$

By splitting the commutator according to (A.10), we can assemble the covariant derivative in the l.h.s. as

$$\nabla_\mu P_\nu^\mu = \nabla_\mu \text{Re}(P_\nu^\mu) + \frac{i}{2} \nabla_\mu \nabla_\nu \mathcal{J}^\mu = \nabla_\mu \text{Re}(P_\nu^\mu) + \frac{i}{2} \mathcal{J}^\mu R_{\mu\nu} + im\partial_\nu \mathcal{P},$$

leaving on the r.h.s a remainder,  $e_\nu^b[\omega_{cab} P_c^a + e\mathcal{J}^a F_{ab} + gj^a U_{ab} + (i/2)\mathcal{J}^a R_{ab}]$ . By virtue of Eqs.(B.4) and (A.6) the imaginary terms on both sides exactly cancel each other and the remaining real part reads as

$$\nabla_\mu \text{Re}(P_\nu^\mu) = e\mathcal{J}^\mu F_{\mu\nu} + gj^\mu U_{\mu\nu} + im[\psi^+ \rho_2 \vec{D}_\nu \psi - \psi^+ \overleftarrow{D}_\nu^+ \rho_2 \psi]. \quad (\text{B.10})$$

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The flux of momentum in the spacelike direction is determined by the Lorentz force that acts on the *electric*



axial current  $e\mathcal{J}^\mu$ , as well as by the Lorentz force of the field  $\aleph_\mu$  that acts on the vector current  $g\mathcal{J}^\mu$ . The last term is due to convection transport of the pseudoscalar mass density  $m\mathcal{P}$  in an electromagnetic field. An importance of axial electric forces for the processes of pion electro-production was noticed by Nambu and Shrauner [25]. The convection term can be cast as

$$m[\psi^+\rho_2(i\partial_b\psi) - (i\partial_b\psi^+)\rho_2\psi + 2eA_b\mathcal{P} + (1/2)\omega_{cdb}\dot{\mathcal{M}}^{*cd}],$$

which repeats the familiar pattern of Gordon's decomposition of the Dirac (vector) current with the replacement  $\rho_1 \rightarrow \rho_2$ ,  $e \rightarrow m$ , and where the long derivative includes *only the electromagnetic potential*. The pseudoscalar density  $\mathcal{P}$  is one of many polarization degrees of freedom of the Dirac field and is not an independent field. In some approximation, the charged pseudoscalar flux inside Dirac waveforms (e.g., nuclei) can be viewed as the interaction of highly localized nucleons via soft pion exchange. Complementary positions of vector and axial currents in Eq.(B.10) prompt a parallel between the charge and chirality of the Dirac field. This parallel was a point of departure for the model of elementary particles developed by Nambu and Jona-Lasinio [26].

Eq.(B.10) also describes a process within a compact object that takes place during a period of its acceleration. In its course, the object changes configuration, undergoes Lorentz contraction and, in fact, becomes a different object, with a larger density and slower flowing time in its interior. Therefore, the unpleasant non-unitarity of the proper Lorentz transformations is a physical effect.

## 2. Pseudoscalar field and $\pi^0 \leftrightarrow 2\gamma$ decay.

The fluxes of charge, mass and momentum carried by the pseudoscalar density in the interaction between Dirac nucleons are commonly attributed to the pion field. Pions and kaons can also be detected as sufficiently long lived particles. Obviously,  $\mathcal{P}$  should satisfy the Klein-Gordon equation, which will be *derived* below as an identity that follows from the Dirac equation. Quite unexpectedly, the positive  $M^2$  term in the Klein-Gordon operator,  $(\square + M^2)$ , will come up as the *negative* scalar Riemannian curvature of the matter-induced metric. This observation explains the puzzling fact that pions and kaons, despite being so narrow long lived resonances, are so easily created in various processes – the negative curvature is typical for the geometry of the expanding matter<sup>15</sup>. Onset of the localization is connected with the instability of the homogeneous expansion.

<sup>15</sup> The pions (and mostly pions) are abundantly created in the high-energy processes where strong contraction of the colliding particles or the small size of initial state is translated into the rapidity plateau in the distribution of pions. About 65% of the decays of the smallest  $\tau$ -lepton go into  $\nu_\tau$  and into one to six pions or kaons. In natural geometry of the expanding matter, at

The first step is to put the axial current in a form with separated convection and polarization currents, as is done in Gordon's decomposition of the vector current,

$$\mathcal{J}^a = -\frac{1}{2m}\eta_{(a)}D_a\mathcal{P} + \frac{1}{2m}I^a; \quad (\text{B.11})$$

$$I^a = -\frac{1}{2}[\psi^+\alpha^{[a}\rho_2\alpha^{b]}\vec{D}_b\psi - \psi^+\overleftarrow{D}_b^+\alpha^{[a}\rho_2\alpha^{b]}\psi].$$

where  $\alpha^{[a}\dots\alpha^{b]}$  stands for  $\alpha^a\dots\alpha^b - \alpha^b\dots\alpha^a$ . (Now, the entire convection term is reduced to the derivative of  $\mathcal{P}$ !) Computing the covariant derivative of both sides of the last equation and using (3.3) we obtain

$$D_a^2\mathcal{P} = 2\psi^+\overleftarrow{D}_a^+\rho_2\vec{D}_a\psi - 2m^2\mathcal{P} - \text{Re}[\psi^+\alpha^{[a}\rho_2\alpha^{b]}(D_aD_b - D_bD_a)\psi]. \quad (\text{B.12})$$

For the stable Dirac field of a nucleon with a large mass  $m$ , we may take the Dirac field in semi-classical approximation,  $\psi \propto e^{iS/\hbar}$ , so that the first two terms in the r.h.s. constitute the classical Hamilton-Jacobi equation for the eikonal  $S$ <sup>16</sup>. If this equation is satisfied with a sufficient accuracy (the waveform behaves as a classical particle and the resonance is sufficiently narrow), then in the r.h.s. remains only the last term. By virtue of Eqs.(A.10) and (A.11) this term becomes nothing but  $eF_{ab}\dot{\mathcal{M}}^{ab} + (R_s/2)\mathcal{P} - (1/2)C_{cab}\psi^+\alpha^a\rho_2\alpha^b\vec{D}_c\psi$ , where  $R_s = R[\psi^+, \psi]$  is the scalar Riemannian curvature (with dimension  $m^2$ ), which is a functional of the Dirac waveform. As a result, we arrive at the *independent* wave equation for the pseudoscalar density  $\mathcal{P}$  (the pion field),

$$D_a^2\mathcal{P} - \frac{R_s}{2}\mathcal{P} \approx -C_{cab}\text{Re}[\psi^+\alpha^a\rho_2\alpha^b\vec{D}_c\psi] + eF_{ab}\dot{\mathcal{M}}^{ab}, \quad (\text{B.13})$$

a Heisenberg equation of motion with a *variable mass* defined by the *negative* scalar Riemannian curvature  $R_s$  outside the stable nucleons. Quite surprisingly, exactly *this*  $\mathcal{P}$  enters the r.h.s. of Eq.(3.18) that defines the force of gravity/inertia in the same approximation of a material point.

The source in the r.h.s. is Hermitian. Its first term is “geometric” and accounts for the flux of momentum

any given moment of time  $t$ , the proper time at a distance  $x$  from a generic point  $x = 0$  corresponds to the earlier proper time  $\tau$  and has a larger density  $\mathcal{R}(x, t)$ . Therefore, as can be perceived from any point, the proper time flows more slowly with larger distance  $x$  from this point. As a result, Dirac waves tend to refract into distant spatial regions. The issue of instability of a uniform expansion of the Dirac field and, thus, of the *dynamically generated charge asymmetry* (and, eventually, of the baryonic asymmetry [18]) of its spontaneous localization, will be discussed elsewhere.

<sup>16</sup> In the semi-classical approximation, the Hamilton-Jacobi equation,  $g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 = 0$  is nothing but the condition that the determinant of the Dirac equation is zero,  $\det[i\gamma^\mu\partial_\mu S + m] = 0$ .

and twist of the tetrad basis. It vanishes when normal coordinates can be introduced. The second term is more related to the pion's dynamics and decay and it can be rewritten as

$$2e\psi^+[\rho_1\vec{E} + \rho_2\vec{B}] \cdot \vec{\sigma}\psi = 2e[\vec{L} \cdot \vec{E} + \vec{K} \cdot \vec{B}]$$

– electric field interacts with magnetic polarization  $\vec{L}$  of the Dirac field and magnetic field interacts with the electric polarization  $\vec{K}$  (cf. Eq.(2.6)). When  $F_{ab}$  is the field of a standing transverse electromagnetic wave this term has a simple representation in terms of the spin interaction with two waves of circular polarizations,

$$eF_{ab}\mathcal{M}^{ab} = 4e \sum_k \frac{-i\sqrt{\omega_k}}{\sqrt{2(2\pi)^3}} (C_k e^{-ikx} - C_k^* e^{ikx}) \times [\vec{e}_L \cdot \vec{\psi}_R \vec{\sigma} \psi_L + \vec{e}_R \cdot \vec{\psi}_L \vec{\sigma} \psi_R], \quad (\text{B.14})$$

where  $\vec{e}_{L,R}(k) \perp \vec{k}$  are the vectors of the two circular polarizations,  $\psi_{L,R}$  are the left and right components of the Dirac spinor field,  $\omega_k = E_\gamma$  is the “photon's energy” and  $C_k$  is the Fourier component of the initial or final (possibly, coherent) state of the Heisenberg field  $F_{ab}$ . This form of the source of the pion field allows one to qualify  $\pi^0$  as a resonance in the system of the Dirac field and a standing electromagnetic wave formed by the two circular polarizations, which causes the simultaneous flip of helicity of both components of the Dirac field. On the other hand the source in the wave equation (B.13) has the structure of the axial anomaly. This could be an exact correspondence if there was a simple proportionality between  $\mathcal{M}^{ab}$  and  $F^{ab}$ . Then,  $eF_{ab}\mathcal{M}^{ab} = CeF_{ab}F^{ab}$ , where the explicit value of  $C$  must comply with the observed rate of the  $\pi^0 \rightarrow 2\gamma$  decay. It is instructive that the wave equation for the pseudoscalar meson field  $\mathcal{P}$  (that yields the

pole in the pion propagator) was derived exactly from the original equation Eq.(3.3). The term, which was *ad hoc* added to this equation by S. Adler [30] (in order to save the Ward identity for the axial vertex in triangle graph) has naturally appeared as the source in the wave equation. [The term  $eF_{ab}\mathcal{M}^{ab}$  is readily incorporated into an effective Lagrangian and it allows one to obtain (without resorting to PCAC hypothesis) the known expression for the  $\pi^0 \rightarrow 2\gamma$  rate in the lowest order with respect to electromagnetic interaction.]

In the first approximation, the value of mass of the Dirac field is not important. For this particular resonance, the mass term in the l.h.s. of Eq.(B.13) can be confidently identified with and measured as  $(2E_\gamma)^2$ . In fact,  $m_\pi^2 \sim -R_s/2$  is an independent of the Dirac mass  $m$  measure of the “metric elasticity” in the ground state of the Dirac field, when balance between left and right is probed by electromagnetic field, thus being a fundamental constant. The dynamic quantity  $m_\pi$  is meaningful only for  $\approx 10^{-16}s$  of the resonance spike of the pseudoscalar density. The geometry of currents inside  $\pi^0$  as a finite-sized object is not clear so far. It decays due to tensor polarization currents (eventually producing two photons with the same, left or right, polarization).

The totally dynamic origin of the pion mass term, which is determined by the curvature  $-R_s[\psi^+, \psi]$  of the matter-induced metric, possibly, explains the diversity of faces that pions may reveal in different situations. This variety ranges from soft pion glue within nuclei (when DIS cannot resolve pion's structure functions) and up to free propagation of the massive pions (tracks) at distances that allow for the pion interferometry (sensitive to the microscopic dynamics of pions emission [31]).

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### Appendix C: Angular variables in Dirac equation.

Let us examine the properties of the solutions of the Dirac equation (5.3), neglecting nonlinear terms, in the presence of the radial field  $2g\aleph_r = -\partial_r \Upsilon$  and in a perfectly spherically symmetric geometry. Since we will focus on the nature of angular variables, an explicit dependence  $\aleph_r(r)$  is not essential. [One may think of Eq.(4.17) as an example.] The only non-vanishing components of the Ricci rotation coefficients are  $\omega_{212} = (1/r) \cot \theta$  and  $\omega_{131} = \omega_{232} = 1/r$ . Solely as a reference, assume that there is an external electromagnetic Coulomb field  $A_0(r)$ . Then the Dirac equation is

$$[i\partial_0 - eA_0 + g\sigma_3\aleph_r - i\rho_3\sigma_3(\partial_r + \frac{1}{r}) - i\frac{\rho_3\sigma_1}{r}(\partial_\theta + \frac{1}{2}\cot\theta) - i\frac{\rho_3\sigma_2}{r\sin\theta}\partial_\varphi - m\rho_1]\psi = 0. \quad (\text{C.1})$$

In terms of a new unknown function,  $\tilde{\psi}(r, \theta, \varphi) = r\sqrt{\sin\theta}\psi$ , this equation becomes

$$[i\partial_0 - eA_0 + g\sigma_3\aleph_r + \rho_3\sigma_3(-i\partial_r) + \frac{\rho_3}{r}(-i\sigma_1\partial_\theta - i\frac{\sigma_2}{\sin\theta}\partial_\varphi) - m\rho_1]\tilde{\psi} = 0. \quad (\text{C.2})$$


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Equation (C.1) is the Dirac equation in the tetrad basis. In order to find its solution one has to separate the angular and radial variables. This is known to be a some-

what tricky problem, even in the standard problem with a radial Coulomb field (when  $\aleph_a = 0$ ). The Hermitian operators in Eq. (C.2) are the tetrad components of the

momenta  $p_3 = -i\partial_r$ ,  $p_1 = -ir^{-1}\partial_\theta$ ,  $p_2 = -i(r\sin\theta)^{-1}\partial_\varphi$ . The operators  $p_1$  and  $p_2$  are clearly associated with the angular motion. If the coefficients in this equation were not matrices, it would have already been an equation with separated variables, which would match the perfect spherical symmetry of the *external* fields  $A_0$  and  $\aleph_r$ . The problem is that the operators of the radial and angular momenta do not commute (they anti-commute,  $[\alpha_3 p_3, (\alpha_1 p_1 + \alpha_2 p_2)]_+ = 0$ ). A regular way to avoid this obstacle is as follows [32, 33]: One attempts to construct a minimal set of operators that commute with the Hamiltonian. For example, one can check that the commutator  $[\alpha_3 p_3, \rho_1(\alpha_1 p_1 + \alpha_2 p_2)]_- = 0$  and take the operator  $\rho_1(\alpha_1 p_1 + \alpha_2 p_2)$  as a generator of the conserved quantum number. This trick works when  $\aleph = 0$  and it is very instructive to see the details of its failure when  $\aleph \neq 0$ .

The conventional operator of angular momentum is  $\vec{\mathcal{L}} = [\vec{r} \times \vec{p}] + \vec{\sigma}/2$ . An additional operator  $\mathcal{L} = \vec{\sigma} \cdot \vec{\mathcal{L}} - 1/2$  commutes with the orbital momentum,  $[\mathcal{L}, (\vec{r} \times \vec{p})] = 0$ , and has the properties,  $\mathcal{L}(\mathcal{L} - 1) = [\vec{r} \times \vec{p}]^2$  and  $\mathcal{L}^2 = \vec{\mathcal{L}}^2 + 1/4$ . Therefore, if  $\kappa$  is an eigenvalue of operator  $\mathcal{L}$  we obviously have  $\kappa(\kappa - 1) = l(l + 1)$  and  $\kappa^2 > 0$ . On the other hand, if  $\mathcal{L}_A = \rho_1 \mathcal{L}$ , then  $(\mathcal{L}_A)^2 = \mathcal{L}^2$  and these operators have the same sets of eigenvalues. In the tetrad basis, these generators of the angular quantum numbers are

$$\mathcal{L}_A = \rho_1 \left( -i\sigma_2 \partial_\theta + \frac{i\sigma_1}{\sin\theta} \partial_\varphi \right), \quad \mathcal{L}_3 = -i\partial_\varphi + \frac{1}{2}\sigma_3. \quad (\text{C.3})$$

In terms of the *auxiliary* operator  $\mathcal{L}_A$ , which has the same set of quantum numbers as the operator of the angular momentum but is a different operator (associated with the tetrad components of the magnetic part  $\vec{L}$  of the tensor  $\mathcal{M}^{ab}$ ), and the projection  $\mathcal{L}_3$  of angular momentum, the Dirac equation becomes

$$[i\partial_0 - eA_0 + g\sigma_3 \aleph_r + \rho_3 \sigma_3 (-i\partial_r) - \rho_2 \sigma_3 \frac{\mathcal{L}_A}{r} - m\rho_1] \tilde{\psi} = 0. \quad (\text{C.4})$$

When the operator  $\mathcal{L}_A$  commutes with all terms of Hamiltonian (which is the case when  $\aleph_\mu = 0$ ) we can require the wave function be an eigenfunction of the Hamiltonian and these two operators,

$$\mathcal{L}_A \tilde{\psi} = \kappa \tilde{\psi}, \quad \text{and} \quad \mathcal{L}_3 \tilde{\psi} = (m_3 + \frac{1}{2}) \tilde{\psi}. \quad (\text{C.5})$$

Even in this case the conserved quantum number  $\kappa$  belongs to the magnetic polarization  $\vec{L}$ , which mixes right and left components of the Dirac spinor, and not to the geometric angular momentum, which does not do that! Since the  $\mathcal{L}_A$  anti-commutes with  $g\sigma_3 \aleph_r$ , we have no obvious solution for the separation of variables. Parallel transport mixes the rotation-like  $\vec{L}$ , which are supposed to be the generators of displacement along angular arcs

(2.7), with the boost-like electric polarization  $\vec{K}$ . These operators do not commute with the Hamiltonian and there indeed may even be no meaningful holonomic coordinates associated with these arcs.

Because the presence of the component  $\aleph_r(r)$  at least apparently preserves the spherical symmetry, we can try to look for a general solution of the following form ( $\xi$  and  $\eta$  are the left and right components of the Dirac field in spinor representation, respectively),

$$\tilde{\xi} = \begin{pmatrix} u_L(r, t) \mathcal{Y}(\theta, \varphi) \\ d_L(r, t) \mathcal{Z}(\theta, \varphi) \end{pmatrix}, \quad \tilde{\eta} = \begin{pmatrix} u_R(r, t) \mathcal{Y}(\theta, \varphi) \\ d_R(r, t) \mathcal{Z}(\theta, \varphi) \end{pmatrix}. \quad (\text{C.6})$$

As a first step, we may try to substitute the Dirac spinor (C.6) into Eqs.(C.5). One can immediately see that the angular variables in (C.5) can be separated only when  $u_L = d_R$  and  $u_R = d_L$ . At the same time, by inspection of the complete system of four Dirac equations,

$$\begin{aligned} (i\partial_0 - eA_0 - g\aleph_r - i\partial_r)u_L \mathcal{Y} &= mu_R \mathcal{Y} + d_L \frac{i\Lambda_-}{r} \mathcal{Z}, \\ (i\partial_0 - eA_0 + g\aleph_r + i\partial_r)d_L \mathcal{Z} &= md_R \mathcal{Z} + u_L \frac{i\Lambda_+}{r} \mathcal{Y}, \\ (i\partial_0 - eA_0 - g\aleph_r + i\partial_r)u_R \mathcal{Y} &= mu_L \mathcal{Y} - d_R \frac{i\Lambda_-}{r} \mathcal{Z}, \\ (i\partial_0 - eA_0 + g\aleph_r - i\partial_r)d_R \mathcal{Z} &= md_L \mathcal{Z} - u_R \frac{i\Lambda_+}{r} \mathcal{Y}, \end{aligned} \quad (\text{C.7})$$

where

$$\Lambda_\pm = (\partial_\theta \pm \frac{i}{\sin\theta} \partial_\varphi), \quad (\text{C.8})$$

one can see that the condition  $u_L = d_R$  and  $u_R = d_L$  is inconsistent with the presence of the field  $\aleph_a$ . The Dirac equation breaks up into two systems of equations for only two radial functions which are incompatible unless  $\aleph_r = 0$ .

Nevertheless, just by inspection, one can see that the angular functions  $\mathcal{Y}_{k,m}(\theta, \varphi)$  and  $\mathcal{Z}_{k,m}(\theta, \varphi)$  that satisfy the equations,

$$\begin{aligned} \Lambda_- \mathcal{Z}_{k,m}(\theta, \varphi) &= -k \mathcal{Y}_{k,m}(\theta, \varphi), \\ \Lambda_+ \mathcal{Y}_{k,m}(\theta, \varphi) &= k \mathcal{Z}_{k,m}(\theta, \varphi), \end{aligned} \quad (\text{C.9})$$

do separate the angular variables in the Dirac equation (and do not separate them in Eq. (C.5)). With this separation of angular variables, Eqs. (C.7) yield a system of four differential equations for the four radial functions. In general, this system is nonlinear and it is not readily split into a system of two second order differential equations, so that the entire problem of the states with negative energy may look differently. Eqs. (C.9) clearly are the equations for the spherical harmonics but  $\theta$  and  $\varphi$  are not the angles of the spatial angular coordinates.

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